

RENORMALIZATION GROUP IMPROVED GRAVITATIONAL ACTIONS: A BRANS-DICKE APPROACH

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Abstract

A new framework for exploiting information about the renormalization group (RG) behavior of gravity in a dynamical context is discussed. The Einstein-Hilbert action is RG-improved by replacing Newton's constant and the cosmological constant by scalar functions in the corresponding Lagrangian density. The position dependence of G and Λ is governed by a RG equation together with an appropriate identification of RG scales with points in spacetime. The dynamics of the fields G and Λ does not admit a Lagrangian description in general. Within the Lagrangian formalism for the gravitational field they have the status of externally prescribed "background" fields. The metric satisfies an effective Einstein equation similar to that of Brans-Dicke theory. Its consistency imposes severe constraints on allowed backgrounds. In the new RG-framework, G and Λ carry energy and momentum. It is tested in the setting of homogeneous-isotropic cosmology and is compared to alternative approaches where the fields G and Λ do not carry gravitating 4-momentum. The fixed point regime of the underlying RG flow is studied in detail.

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1 Introduction

It is an old idea that Newton’s “constant” actually is not really constant throughout spacetime but varies from one spacetime point to another. The perhaps most popular self-consistent theory of gravity which implements this idea is Brans-Dicke theory [1]. Originally devised in an attempt at modifying General Relativity so as to become compatible with Mach’s principle, it is by now the prototype of a theory in which the gravitational interaction is mediated by the metric together with some additional non-geometric field, here a scalar ϕ . The Brans-Dicke field ϕ is introduced as the inverse of the position dependent Newton constant: $\phi(x) \equiv 1/G(x)$. In the original form of the theory its dynamics, along with that of the metric, is derived from the action

$$S_{\text{BD}} = \frac{1}{16\pi} \int d^4x \sqrt{-g} (\phi R - \omega \phi^{-1} \partial_\mu \phi \partial^\mu \phi) + S_{\text{M}}. \quad (1.1)$$

It supplements the Einstein-Hilbert term with an (almost) conventional scalar kinetic term. (S_{M} denotes the matter action.) Varying S_{BD} with respect to the metric $g_{\mu\nu}$ and ϕ leads, respectively, to a modified Einstein equation,

$$G_{\mu\nu} = 8\pi \phi^{-1} (T_{\mu\nu} + \mathcal{T}_{\mu\nu}^\omega) + \phi^{-1} (D_\mu D_\nu \phi - g_{\mu\nu} D^2 \phi), \quad (1.2)$$

and the scalar equation of motion

$$D^2 \phi = \frac{8\pi}{3 + 2\omega} T_\mu{}^\mu. \quad (1.3)$$

Here $T_{\mu\nu}$ is the energy-momentum tensor obtained from S_{M} , and

$$\mathcal{T}_{\mu\nu}^\omega = \frac{\omega}{8\pi \phi} \left[D_\mu \phi D_\nu \phi - \frac{1}{2} g_{\mu\nu} D_\rho \phi D^\rho \phi \right] \quad (1.4)$$

is the one stemming from the kinetic term of ϕ ; its normalization ω is a free parameter a priori. Together with the $\phi^{-1} DD\phi$ -terms in (1.2), originating from the x -dependence of ϕ in the Lagrangian $\sqrt{-g} \phi R$, the tensor $\mathcal{T}_{\mu\nu}^\omega$ describes the 4-momentum carried by the scalar field ϕ . The point to be emphasized here is that in Brans-Dicke theory the

x -dependence of Newton's constant is governed by a simple local equation of motion with an obvious Lagrangian or Hamiltonian formulation, the Klein-Gordon equation (1.3).

Recently a lot of work was devoted to a different type of theories with a variable Newton constant where the dynamics of $G(x)$ does not admit a straightforward Lagrangian description and simple local equations of motion such as (1.3). These theories arise by “renormalization group (RG) improving” classical General Relativity, i. e. by replacing G and similar generalized couplings such as the cosmological constant Λ by scale dependent or “running” quantities [2]. Here the starting point is a scale dependent effective action for the gravitational field, $\Gamma_k[g_{\mu\nu}]$, a Wilson-type (“coarse-grained”) free energy functional. It defines an effective field theory valid near the mass scale k or length scale $\ell \equiv k^{-1}$. This means that in order to include all fluctuation effects relevant at k it is sufficient to employ Γ_k at tree level. In the context of quantum field theory, Γ_k obtains from the fundamental (bare) action of the theory by integrating out all quantum fluctuations with momenta larger than the infrared cutoff k , i. e. wavelengths smaller than ℓ . The “effective average action” [3, 4] is a concrete realization of this idea; following similar lines as in Yang-Mills theory [5] it has been applied to the quantized gravitational field [6] and to quantum gravity with matter [7]. Another logical possibility is that the scale-dependence or “running” of Γ_k arises as the result of a purely classical averaging process [8].

In either case the k -dependence of Γ_k is governed by a functional differential equation, the “flow equation” or “exact RG equation” [9]. A general functional $\Gamma_k[g_{\mu\nu}]$ can be parameterized by infinitely many dimensionless couplings related to the coefficients of the field monomials (higher powers of the curvature, non-local terms [10], etc.). They include the dimensionless Newton constant $g(k) \equiv k^2 G(k)$ and cosmological constant $\lambda(k) \equiv \Lambda(k)/k^2$. When expressed in terms of the running coupling constants the flow equation assumes the form of a system of infinitely many ordinary coupled differential

equations:

$$\begin{aligned}
k \frac{d}{dk} \lambda(k) &= \beta_\lambda(\lambda, g, \dots) \\
k \frac{d}{dk} g(k) &= \beta_g(\lambda, g, \dots) \\
&\vdots
\end{aligned} \tag{1.5}$$

A solution $k \mapsto (\lambda(k), g(k), \dots)$ corresponds to a RG trajectory on “theory space”, the space of all action functionals. Because of the technical complexity of the problem one is often forced to restrict the RG flow to a finite-dimensional subspace, a truncated theory space. In the “Einstein-Hilbert truncation”, say, only the couplings g and λ are taken into account; every solution $(\lambda(k), g(k))$ of the truncated flow equation corresponds to the one-parameter family of action functionals

$$\Gamma_k[g_{\mu\nu}] = \frac{1}{16\pi G(k)} \int d^4x \sqrt{-g} [R - 2\Lambda(k)] \tag{1.6}$$

with G and Λ given by

$$G(k) = g(k)/k^2 \quad \text{and} \quad \Lambda(k) = \lambda(k) k^2. \tag{1.7}$$

Accidentally, in this truncation the resulting effective field equations (with a matter term added) look like the standard Einstein equation:

$$G_{\mu\nu} = -\Lambda(k) g_{\mu\nu} + 8\pi G(k) T_{\mu\nu}. \tag{1.8}$$

For Quantum Einstein Gravity, truncated flow equations were derived in [6], [7], and [11] - [15], for instance. In ref. [15] a nonlocal truncation ansatz had been used [16].

Once the k -dependence of G , Λ , and the other parameters is known for the RG trajectory of interest one can try to use this information in order to “RG improve” the predictions of standard General Relativity. One looks for a “cutoff identification” of the form [17]

$$k = k(x) \tag{1.9}$$

which would convert the scale-dependence of G, Λ, \dots to a position-dependence:

$$G(x) = G(k(x)), \quad \Lambda(x) = \Lambda(k(x)), \quad \dots \quad (1.10)$$

It is certainly not always possible to associate cutoff values to spacetime points and to find an appropriate function $k(x)$. However, in physical situations with a high degree of symmetry, very often symmetry arguments and dimensional analysis lead to essentially unique answers for $k(x)$. Let us mention two examples where a natural and physically transparent cutoff identification suggests itself.

We consider the cosmology of a homogeneous and isotropic Universe and assume that the metric has been brought to the standard Robertson-Walker form with a scale factor $a(t)$. Then the postulate of homogeneity and isotropy implies that k can depend on the cosmological time only, either explicitly or implicitly via the scale factor: $k = k(t, a(t), \dot{a}(t), \dots)$. In ref. [18] we gave detailed arguments as to why the purely explicit time dependence

$$k(t) = \xi/t \quad (1.11)$$

with ξ a positive constant of order unity is the correct identification in a first approximation. The reason is that, when the age of the Universe is t , clearly no quantum or classical fluctuation with a frequency smaller than $1/t$ can have played any role yet. Hence the integrating-out of modes or “coarse-graining” which underlies the Wilson renormalization group should be stopped at $k \approx 1/t$. Moreover, for many cosmologies of interest other plausible cutoffs such as $k = H \equiv \dot{a}/a$ are equivalent to (1.11). (The cutoff identification (1.11) has been used in refs. [18–20] in order to improve the cosmological evolution *equations* of General Relativity.)

In ref. [21] the RG-improvement of a Schwarzschild black hole, i.e. of a *solution* to the standard Einstein equation has been discussed. Here the symmetries imply that k can be dependent on the Schwarzschild radial coordinate r only. The natural cutoff

identification which can be motivated in various ways [21] is

$$k(r) = \xi/d(r). \quad (1.12)$$

Here $d(r)$ is the proper distance from a point with coordinate r to the center of the black hole. In [21], $d(r)$ was computed from the unperturbed Schwarzschild metric; in a more refined treatment the improved metric should be used instead.

Let us now discuss the physical cutoff mechanism in the generic case where the problem is not highly symmetric with a single preferred scale. Within the effective average action formalism the general rule is to follow the RG flow from the bare action $\Gamma_{k=\infty} \equiv S$, which serves as the initial condition for the Γ_k -flow equation, all the way down to $k = 0$. The endpoint of the RG trajectory thus obtained, $\Gamma_{k=0} \equiv \Gamma$, is nothing but the ordinary effective action [4]. If we have no a priori knowledge about a physical cutoff, what we have to do is to solve the effective equation of motion

$$\frac{\delta\Gamma[g_{\mu\nu}]}{\delta g_{\mu\nu}} = 0. \quad (1.13)$$

It is well-known that, despite its classical appearance, this equation is fully quantum mechanical, and metrics $g_{\mu\nu} \equiv \langle g_{\mu\nu} \rangle$ satisfying it are exact quantum vacuum solutions.

In practice, in any realistic theory, it is impossible to compute the RG trajectories exactly. One is forced to truncate the theory space, and very often the truncations used are reliable only for large values of k (UV) but not for small ones (IR). Typically, and in particular for asymptotically free theories, simple local truncations are sufficient in the UV, but for $k \rightarrow 0$ nonlocal terms must be included in the truncation ansatz for Γ_k . In particular in massless theories it is technically extremely difficult to handle those nonlocal terms. In QCD, say, a reliable calculation of Γ_k for $k \rightarrow 0$ is still out of reach, and the corresponding problems in Quantum Gravity are much harder even [12, 15]. While $\Gamma_{k \rightarrow 0}$ is not available in full generality, the method outlined above (inserting $k = k(x)$ into a fairly simple truncated RG trajectory) is a kind of “short-cut” for the way from the

UV to the IR. Even though we use a strictly local truncation, the cutoff identification $k = k(x)$ introduces nonlocal features into the theory which, under certain conditions, are equivalent to some of the elusive nonlocal terms in Γ_k or the Γ of the standard approach. In order to explain this partial equivalence we must review the phenomenon of *decoupling* in the formalism of the effective average action [3, 4].

For the sake of simplicity let us discuss the universality class of a single real, Z_2 -symmetric scalar field $\Phi(x)$ in flat Euclidean space. Its effective average action $\Gamma_k[\Phi]$ is to be determined from the flow equation [3]

$$k \frac{d}{dk} \Gamma_k[\Phi] = \frac{1}{2} \text{Tr} \left[\left(\Gamma_k^{(2)} + R_k \right)^{-1} k \frac{d}{dk} R_k \right] \quad (1.14)$$

subject to the initial condition $\Gamma_\infty = S$. Here $\Gamma_k^{(2)}$ denotes the infinite dimensional matrix of second functional derivatives of $\Gamma_k[\Phi]$, and R_k is the cutoff function. For the purposes of the present argument it is sufficient to employ a R_k of “mass type”: $R_k = k^2$ [4]. An important class of nonperturbative solutions to (1.14) can be found with the ansatz

$$\Gamma_k[\Phi] = \int d^4x \left\{ \frac{1}{2} Z(k) \partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} m^2(k) \Phi^2 + \frac{1}{12} \lambda(k) \Phi^4 + \dots \right\}. \quad (1.15)$$

To start with, we neglect the running of the kinetic term (“local potential ansatz”) and approximate $Z(k) \equiv 1$. For functionals of this type, and in a momentum basis where $-\partial^2 \stackrel{\wedge}{=} p^2$, the denominator appearing under the trace of (1.14) reads

$$\Gamma_k^{(2)} + R_k = p^2 + m^2(k) + k^2 + \lambda(k) \Phi^2 + \dots. \quad (1.16)$$

In a diagrammatic loop calculation of Γ_k it is the inverse of (1.16), evaluated at $\Phi = \langle \Phi \rangle$, which appears as the effective propagator in all loops. It contains an IR cutoff at the scale k , a simple mass term k^2 which adds to $m^2(k)$ in the special case considered here. (In general $R_k \equiv R_k(p^2)$ introduces a p^2 -dependent mass.) The p_μ -modes (plane waves) are integrated out efficiently only in the domain $p^2 \gtrsim m^2 + k^2 + \lambda \Phi^2 + \dots$. In the opposite case all loop contributions are suppressed by the effective mass square $m^2 + k^2 + \lambda \Phi^2 + \dots$.

It is the sum of the “artificial” cutoff k^2 , introduced in order to effect the coarse graining, and the “physical” cutoff terms $m^2(k) + \lambda(k) \Phi^2 + \dots$. As a consequence, Γ_k displays a significant dependence on k only if $k^2 \gtrsim m^2(k) + \lambda(k) \Phi^2 + \dots$ because otherwise k^2 is negligible relative to $m^2 + \lambda \Phi^2 + \dots$ in all propagators; it is then the physical cutoff scale $m^2 + \lambda \Phi^2 + \dots$ which delimits the range of p^2 -values which are integrated out. Typically, for k very large, k^2 is larger than the physical cutoffs so that Γ_k “runs” very fast. Lowering k it might happen that, at some $k = k_{\text{dec}}$, the “artificial” cutoff k becomes smaller than the running mass $m(k)$. At this point the physical mass starts playing the role of the actual cutoff; its effect overrides that of k so that Γ_k becomes approximately independent of k for $k < k_{\text{dec}}$. As a result, $\Gamma_k \approx \Gamma_{k_{\text{dec}}}$ for all k below the threshold k_{dec} , and in particular the ordinary effective action $\Gamma = \Gamma_0$ does not differ from $\Gamma_{k_{\text{dec}}}$ significantly. This is the prototype of a “decoupling” or “threshold” phenomenon.

The situation is more interesting when m^2 is negligible and k^2 competes with $\lambda \Phi^2$ for the role of the actual cutoff. (Here we assume that Φ is x -independent.) The running of Γ_k , evaluated at a fixed Φ , stops once $k \lesssim k_{\text{dec}}(\Phi)$ where the by now field dependent decoupling scale obtains from the implicit equation $k_{\text{dec}}^2 = \lambda(k_{\text{dec}}) \Phi^2$. Decoupling occurs for sufficiently large values of Φ , the RG evolution below k_{dec} is negligible then; hence, at $k = 0$,

$$\Gamma[\Phi] = \Gamma_k[\Phi] \Big|_{k=k_{\text{dec}}(\Phi)}. \quad (1.17)$$

Eq. (1.17) is an extremely useful tool for effectively going beyond the truncation (1.15) without having to derive and solve a more complicated flow equation. In fact, thanks to the additional Φ -dependence which comes into play via $k_{\text{dec}}(\Phi)$, eq. (1.17) can predict certain terms which are contained in Γ even though they are not present in the truncation ansatz.

A simple example illustrates this point. For k large, the truncation (1.15) yields a logarithmic running of the Φ^4 -coupling: $\lambda(k) \propto \ln(k)$. As a result, (1.17) suggests

that Γ should contain a term $\propto \ln(k_{\text{dec}}(\Phi)) \Phi^4$. Since, in leading order, $k_{\text{dec}} \propto \Phi$, this leads us to the prediction of a $\Phi^4 \ln(\Phi)$ -term in the conventional effective action. This prediction, including the prefactor of the term, is known to be correct actually: the Coleman-Weinberg potential of massless Φ^4 -theory does indeed contain a $\Phi^4 \ln(\Phi)$ -term. Note that this term admits no power series expansion in Φ , so it lies outside the space of functionals spanned by the original ansatz (1.15).

This example nicely illustrates the power of the decoupling arguments. They can be applied even when Φ is taken x -dependent as it is necessary for computing n -point functions by differentiating $\Gamma_k[\Phi]$. The running inverse propagator is given by $\Gamma_k^{(2)}(x - y) = \delta^2 \Gamma_k / \delta \Phi(x) \delta \Phi(y)$, for example. Here a new potential cutoff scale enters the game: the momentum q dual to the distance $x - y$. When it serves as the acting IR cutoff, the running of $\tilde{\Gamma}_k^{(2)}(q)$, the Fourier transform of $\Gamma_k^{(2)}(x - y)$, stops once k^2 is smaller than $k_{\text{dec}}^2 = q^2$. Hence $\tilde{\Gamma}_k^{(2)}(q) \approx \tilde{\Gamma}_k^{(2)}(q) \Big|_{k=\sqrt{q^2}}$ for $k^2 \lesssim q^2$, provided no other physical scales intervene. As a result, if one allows for a running Z -factor in the truncation (1.15) one predicts a propagator of the type $\left[Z \left(\sqrt{q^2} \right) q^2 \right]^{-1}$ in the standard effective action. Note that it corresponds to a nonlocal term $\propto \Phi Z \left(\sqrt{-\partial^2} \right) \partial^2 \Phi$ in Γ , even though the truncation ansatz was perfectly local.

At this point also the origin of the x -dependent scale $k = k(x)$ employed in the present paper becomes clear: The distance $|x - y| \equiv r$ translates to a momentum $\sqrt{q^2} \approx 1/r$ so that the running of $Z(k)$ is stopped at the physical scale $1/r$. This is precisely the interpretation we use in Quantum Gravity, with $\sqrt{g} R/G(k)$ taking the place of $Z(k) \left(\partial_\mu \Phi \right)^2$, and (1.12) covariantizing $k_{\text{dec}} \propto 1/r$.

More generally, by inserting $k = k(x)$ into $G(k)$ we try to mimic the effect of terms contained in Γ , but not in the Einstein-Hilbert truncation. In this sense the strategy is similar to the derivation of the $\Phi^4 \ln(\Phi)$ - and the nonlocal kinetic term above.

RG improvement based upon the above ideas on decoupling is a powerful tool, a kind of “short-cut” from the UV to the IR, if one is able to identify the actual physical

cutoff without solving for the full RG flow. This requires a detailed case-by-case study of the physical problem at hand; for cosmology and black holes see refs. [18] and [21] for a justification of (1.11) and (1.12), respectively. The identification of the physical cutoff is greatly facilitated by symmetry arguments and, in particular near a scale invariant fixed point, by dimensional analysis. But even in the generic case it is sometimes possible to get a handle on the essential physics by a careful analysis of the various potential cutoff terms in $\Gamma_k^{(2)}$.

Let us return to gravity now. Generally speaking, every pair consisting of a RG trajectory $(\lambda(k), g(k), \dots)$ and a cutoff identification $k = k(x)$ generates a set of position dependent “constants” $\Lambda(x), G(x), \dots$. Clearly the dynamical origin of the x -dependence of G , say, is rather different from what happens in Brans-Dicke theory. The basic “equation of motion” is the exact RG equation. It is to be solved before $k(x)$ is inserted and has no connection to any specific spacetime therefore, in contradistinction to the Klein-Gordon equation for ϕ . A closely related fact, which will become very important later on and is in fact one of the main motivations for the present paper, is the following: In general there exists no Lagrangian formulation of the system “RG equation plus cutoff identification” which could take the place of the $\phi^{-1} \partial_\mu \phi \partial^\mu \phi$ -term.¹ While one may continue to specify the gravitational dynamics by means of a Einstein-Hilbert-like action, the couplings $\Lambda(x), G(x), \dots$ have the status of externally prescribed fields. In a sense, we are dealing here with a kind of background field problem: the metric $g_{\mu\nu}$ must be determined in presence of the “background fields” $(\Lambda(x), G(x), \dots)$ on which it functionally depends therefore. We shall see that not all backgrounds are admissible and it will be an important question which ones lead to consistent field equations for $g_{\mu\nu}$.

In the rest of this paper we restrict the discussion to the Einstein-Hilbert truncation. Let us assume we have solved its RG equation and obtained $(\Lambda(x), G(x))$ from some RG

¹Of course the exact framework (construct Γ , solve (1.13)) is Lagrangian, but this is of no help if we take the “short-cut” discussed above.

trajectory. How can we take advantage of this information? A possible strategy is to insert the identification $k = k(x)$ into the effective field equation (1.8),

$$G_{\mu\nu} = -\Lambda(x) g_{\mu\nu} + 8\pi G(x) T_{\mu\nu}, \quad (1.18)$$

and to solve this differential equation in presence of the scalar background fields G and Λ . In refs. [18, 19, 22] this approach was employed in the context of cosmology. Some of the corresponding results which we shall need later on are collected in Appendix A.

This strategy, henceforth referred to as the approach of *improving equations*, is not always meaningful. For instance, if we are interested in the structure of black holes in absence of matter ($T_{\mu\nu} = 0$) and at scales where the cosmological constant can be neglected, the improved field equation (1.18) reduces to $G_{\mu\nu} = 0$. This is the standard vacuum Einstein equation which does not know anything about the RG running. Here the leading corrections can be taken into account by *improving solutions* of the classical Einstein equation. The Schwarzschild solution, for instance, satisfies $G_{\mu\nu} = 0$ but it contains G as a constant of integration. The improved Schwarzschild metric is obtained by replacing this constant with $G(x) \equiv G(r)$. (See ref. [21] for a detailed discussion of this metric.)

The approach of improving solutions is much less powerful than that of improving equations. The former is reliable only if the original and improved metrics are not very different, while the latter might well lead to solutions of the improved equations which are quite different from the corresponding classical ones, without necessarily being unreliable [18].

The limited applicability of the “improving solutions” method is one of our motivations for trying to find a new way of injecting the information provided by the renormalization group into the gravitational field equations. The approach we are going to investigate in the present paper is that of *improving actions*. The basic idea is to make the cutoff identification in the action functional, i.e. before the derivation of the field

equations. We start from the k -dependent Einstein-Hilbert action (1.6) and substitute $k = k(x)$ in the corresponding Lagrangian density:

$$S_{\text{mEH}}[g, G, \Lambda] = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left\{ \frac{R}{G(x)} - 2 \frac{\Lambda(x)}{G(x)} \right\}. \quad (1.19)$$

When varied with respect to $g_{\mu\nu}$, this modified Einstein-Hilbert (mEH) functional gives rise to field equations which differ from those of the “improving equations” approach by terms involving derivatives of $G(x)$. The action S_{mEH} is not supposed to be varied with respect to G and Λ . The functions $G(x)$ and $\Lambda(x)$ are obtained from a RG trajectory in the way described above, hence are external to the Lagrangian formalism which we use for the gravitational field. In this sense we are dealing with an “external field Brans-Dicke theory” where the $g_{\mu\nu}$ -dynamics takes place in the fixed background of the scalar fields $G(x)$ and $\Lambda(x)$.

Not all backgrounds lead to a consistent equation for the metric. Both when improving the field equation and the action the resulting modified Einstein equation has the form $G_{\mu\nu} = \mathcal{E}_{\mu\nu}$ where $\mathcal{E}_{\mu\nu}$ is constructed from $T_{\mu\nu}$, G , Λ , and their derivatives. Since $D^\mu G_{\mu\nu} = 0$ by Bianchi’s identity, consistency requires that $D^\mu \mathcal{E}_{\mu\nu} = 0$. In classical General Relativity this condition is satisfied if $T_{\mu\nu}$ is conserved. In the present case additional conditions on $G(x)$ and $\Lambda(x)$ arise, henceforth referred to as “consistency conditions”.

Only if the background satisfies the consistency condition Einstein’s equation can be integrated. However, it should be remarked that in practice this restriction very often is not a drawback but rather an advantage of the formalism. The point to be remembered here is that most features of the RG trajectories are unphysical, i.e. not directly observable, and can be changed by changing the cutoff scheme. The effective average action, for example, has a built-in “shape function” which controls the transition from the momenta which are integrated out to those which are not [6, 11]. Changing the shape function changes the trajectories. Only quantities which, in the language of statistical mechanics, are “universal” remain invariant and thus qualify for the status of

an observable. Therefore, when one realizes that some background does not satisfy the consistency conditions one can try to change the cutoff scheme (shape function) and/or the identification $k = k(x)$ so as to achieve consistency. If we are successful, there is a non-trivial conspiracy between the RG equation and the modified Einstein equation which can teach us something about the correct mathematical model (shape function, cutoff identification) of the physical mechanism which stops the RG running at some scale. (See refs. [18,21] for a detailed discussion of this point.)

Finally let us say a few words about the RG trajectories which we are going to employ in the present paper. They are motivated by the explicit results obtained from the truncated flow equation of Quantum Einstein Gravity [6, 11–15, 23], and by a phenomenologically inspired conjecture formulated in [19].

Within the Einstein-Hilbert truncation of the effective average action, the RG equations for $g(k)$ and $\lambda(k)$ were first derived in [6] and solved numerically in [12]. The RG flow on the g - λ -plane is dominated by two fixed points (g_*, λ_*) : a Gaussian fixed point at $g_* = \lambda_* = 0$, and a non-Gaussian one with $g_* > 0$ and $\lambda_* > 0$ [23]. The high-energy or short-distance behavior of Quantum Einstein Gravity is governed by the non-Gaussian fixed point: for $k \rightarrow \infty$, all RG trajectories run into this fixed point. If it is present in the exact flow equation, it can be used to construct a fundamental, i.e. microscopic quantum theory of gravity by taking the limit of infinite UV cutoff along one of the trajectories running into the fixed point. This corresponds precisely to Weinberg's asymptotic safety scenario [24]: performing the UV limit at a fixed point one can be sure that the theory does not develop uncontrolled singularities at high energies.

In refs. [11,13,14] detailed consistency checks were performed which indicate that the non-Gaussian fixed point should be a reliable prediction of the Einstein-Hilbert truncation, and that Quantum Einstein Gravity has very good chances of being a non-perturbatively renormalizable (and not only an effective) field theory of gravity. A conceptually independent investigation which points in exactly the same direction is the quantization of

the 2 Killing vector-reduction of Einstein gravity in ref. [25].

At the fixed point, the dimensionless couplings assume constant values g_* and λ_* , respectively. Hence the dimensionful ones run according to

$$G(k) = g_*/k^2, \quad \Lambda(k) = \lambda_* k^2. \quad (1.20)$$

In particular, $G(k) \rightarrow 0$ for $k \rightarrow \infty$, so that Newton's constant is an asymptotically free coupling.

At the other end of the energy scale, for $k \rightarrow 0$, the trajectory with vanishing cosmological constant hits the Gaussian fixed point. This is the realm of classical General Relativity where $G(k)$ assumes an approximately k -independent, non-zero value. In this “perturbative regime” [11] the leading quantum corrections can be computed as a power series in k/m_{Pl} where $m_{\text{Pl}} \equiv [G(k=0)]^{-1/2}$ is the Planck mass [6, 11]. For intermediate values of k , in particular at the cross-over from the UV- to the IR-fixed point, the flow equations must be solved numerically [12].

In this paper we are going to RG-improve homogeneous-isotropic cosmologies for which (1.11) is the appropriate cutoff identification. Given the (numerical) solutions for $G(k)$ and $\Lambda(k)$ we may replace k with ξ/t and obtain the corresponding “background” fields $G(t)$ and $\Lambda(t)$. In particular, near a fixed point, we have

$$G(t) = \tilde{g}_* t^2 \quad (1.21a)$$

$$\Lambda(t) = \tilde{\lambda}_* t^{-2} \quad (1.21b)$$

with the constants

$$\tilde{g}_* \equiv g_* \xi^{-2} \quad (1.22a)$$

$$\tilde{\lambda}_* \equiv \lambda_* \xi^2. \quad (1.22b)$$

Since the time dependence (1.21) applies in the vicinity of a UV fixed point it is realized in the very *early* Universe, i. e. for $t \rightarrow 0$.

In [19] it has been conjectured that in the asymptotically *late* Universe, too, G and Λ have a time dependence given by (1.21), with different values of g_* and λ_* though. According to this conjecture, the $t \rightarrow \infty$ behavior of the Universe is governed by a further non-Gaussian fixed point. It does not occur within the Einstein-Hilbert truncation but there are first indications [15, 16] that the inclusion of non-local invariants could have the desired effect. The main motivation for this conjecture is that it explains the approximate equality of vacuum and matter energy density in the present Universe in a very elegant and natural way [19].

The purpose of the present paper is twofold. In the first part we discuss the general theory of RG-improved gravitational actions, and in the second we test and illustrate this approach in the context of cosmology². To be as explicit as possible and to obtain analytic results, most of the time we shall use the “fixed point background” (1.21) as an example³. This allows us to compare the results obtained by “improving actions” with what had been found by “improving equations”. (The latter results are briefly summarized in Appendix A.)

The remaining sections of this paper are organized as follows. In Section 2 we develop the general framework describing the gravitational dynamics in the background of a position-dependent G and Λ which results from RG improving action functionals of the Einstein-Hilbert type. In particular we identify various classes of solutions to the equations of motion (“Class I, II, III”) which enjoy special properties; some of them can be obtained in a very efficient way by means of a Weyl transformation. In Section 3 the dynamical equations and consistency conditions are specialized for the cosmology of homogeneous and isotropic Universes. Then, in Sections 4 - 6, we find solutions to the cosmological evolution equations belonging to Class I, II, and III, respectively. They

²For related work within standard Brans-Dicke theory see [26] and references therein.

³Using a different approach, RG-improved cosmologies with other time dependencies of Λ were investigated in [27]. A general discussion of cosmologies with a time dependent Λ can be found in [28].

illustrate a number of general properties of our approach in a particularly transparent way. In particular they clarify its relation to alternative strategies for the RG improvement of gravity (improvement at the level of the field equations). The results are summarized in Section 7.

Many of the cosmological solutions found in this paper are not yet realistic from the physical point of view. Their importance resides in the fact that they provide us with valuable insights into the general features of the improved action-approach. The application of this approach to more realistic cosmologies (and the relation to quintessence models [29, 30]) will be discussed elsewhere.

2 The general framework

2.1 The modified Einstein equation

Our starting point is the modified Einstein-Hilbert action $S_{\text{mEH}}[g, G, \Lambda]$ of eq. (1.19) which promotes Newton's constant and the cosmological constant to scalar functions on spacetime. In this setting $G(x)$ and $\Lambda(x)$ are arbitrary, externally prescribed functions which are assumed to have no functional dependence on the metric a priori. In fact, the functional derivative of S_{mEH} with respect to $g_{\mu\nu}$ is given by⁴

$$\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{mEH}}[g, G, \Lambda]}{\delta g_{\mu\nu}(x)} = -\frac{1}{8\pi G(x)} (G^{\mu\nu} + g^{\mu\nu} \Lambda - \Delta t^{\mu\nu}), \quad (2.1)$$

with $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ the usual Einstein tensor. The x -dependence of Newton's constant gives rise to the tensor

$$\begin{aligned} \Delta t_{\mu\nu} &\equiv G(x) (D_\mu D_\nu - g_{\mu\nu} D^2) \frac{1}{G(x)} \\ &\equiv \frac{1}{G^2} \left\{ 2 D_\mu G D_\nu G - G D_\mu D_\nu G - g_{\mu\nu} [2 (DG)^2 - G D^2 G] \right\}. \end{aligned} \quad (2.2)$$

⁴Our curvature conventions are $R^\sigma_{\rho\mu\nu} = -\partial_\nu \Gamma^\sigma_{\mu\rho} + \dots$, $R_{\mu\nu} = R^\sigma_{\mu\sigma\nu}$. The metric signature is $(-+++)$. Frequently we abbreviate $(DG)^2 \equiv g^{\mu\nu} D_\mu G D_\nu G$ and $D^2 G \equiv g^{\mu\nu} D_\mu D_\nu G$.

For later use we also note its trace and covariant divergence:

$$\Delta t_\mu{}^\mu = \frac{3}{G^2} [G D^2 G - 2 (DG)^2], \quad (2.3)$$

$$D_\mu \Delta t^{\mu\nu} = \frac{1}{G} D_\mu G (\Delta t^{\mu\nu} - R^{\mu\nu}). \quad (2.4)$$

We introduce an arbitrary set of matter fields $A(x)$ minimally coupled to gravity. Their dynamics is governed by the action $S_M[g, A]$ which gives rise to the energy-momentum tensor of the matter system in the usual way:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_M[g, A]}{\delta g_{\mu\nu}(x)}. \quad (2.5)$$

The action S_M is assumed to be invariant under general coordinate transformation. As a result, $T_{\mu\nu}$ is conserved when A satisfies its equation of motion, $\delta S_M/\delta A = 0$:

$$D_\mu T^{\mu\nu} = 0. \quad (2.6)$$

Furthermore, we allow for an action $S_\theta[g, G, \Lambda]$ and a corresponding energy-momentum tensor

$$\theta^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_\theta[g, G, \Lambda]}{\delta g_{\mu\nu}(x)} \quad (2.7)$$

which is supposed to describe the 4-momentum carried by the fields $G(x)$ and $\Lambda(x)$. It vanishes for constant fields therefore. The structure of S_θ and $\theta_{\mu\nu}$ is not completely fixed by general principles. In fact, one of the main topics of the present paper is a detailed discussion of the mathematical consistency requirements constraining the form of $\theta_{\mu\nu}$, and of the physical implications of various choices for $\theta_{\mu\nu}$. The only general properties which are assumed throughout are (1), S_θ is independent of the matter fields, (2), S_θ is invariant under general coordinate transformations provided G and Λ are transformed as scalars, and (3), $\theta_{\mu\nu}$ vanishes for $G, \Lambda = \text{const}$ (otherwise we would modify the ordinary Einstein equation).

Thus the total action for the system under consideration is

$$S_{\text{tot}} = S_{\text{mEH}}[g, G, \Lambda] + S_{\text{M}}[g, A] + S_{\theta}[g, G, \Lambda]. \quad (2.8)$$

Varying S_{tot} with respect to the metric we obtain a modified form of Einstein's equation:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + 8\pi G (T_{\mu\nu} + \Delta T_{\mu\nu} + \theta_{\mu\nu}). \quad (2.9)$$

Frequently we shall write it in the form

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu} + \Delta t_{\mu\nu} + \vartheta_{\mu\nu} \quad (2.10)$$

with the convenient definitions

$$\vartheta_{\mu\nu} \equiv 8\pi G \theta_{\mu\nu}, \quad (2.11)$$

$$\Delta t_{\mu\nu} \equiv 8\pi G \Delta T_{\mu\nu}. \quad (2.12)$$

An equivalent form of the field equation is obtained from (2.10) by contracting with $g^{\mu\nu}$:

$$R_{\mu\nu} = Q_{\mu\nu} + \widetilde{\Delta} t_{\mu\nu} + \widetilde{\vartheta}_{\mu\nu}. \quad (2.13)$$

The source terms on the RHS of (2.13) are given by

$$Q_{\mu\nu} \equiv \Lambda g_{\mu\nu} + 8\pi G (T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T), \quad (2.14)$$

$$\widetilde{\Delta} t_{\mu\nu} \equiv \Delta t_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \Delta t_{\alpha}^{\alpha}, \quad (2.15)$$

$$\widetilde{\vartheta}_{\mu\nu} \equiv \vartheta_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \vartheta_{\alpha}^{\alpha}. \quad (2.16)$$

Here and in the following we write $T \equiv T_{\mu}^{\mu}$ for the trace of the energy-momentum tensor in the matter sector.

Einstein's equation is coupled to the equations of motion of the matter system,

$$\frac{\delta S_{\text{tot}}}{\delta A} = \frac{\delta S_{\text{M}}}{\delta A} = 0. \quad (2.17)$$

We stress that (2.17) and Einstein's equation $\delta S_{\text{tot}}/\delta g_{\mu\nu} = 0$ are the only field equations to be derived from S_{tot} ; there are no analogous equations for G and Λ such as $\delta S_{\text{tot}}/\delta G = 0 = \delta S_{\text{tot}}/\delta \Lambda$. This is a key difference between our approach and standard Brans-Dicke type theories. As we emphasized already, $G(x)$ and $\Lambda(x)$ are *externally prescribed* functions in our case which are determined by the RG equations for $G(k)$ and $\Lambda(k)$ and an appropriate cutoff identification. For this reason the status of G and Λ is different from that of a scalar matter field A . For instance, their energy-momentum tensor $\theta_{\mu\nu}$ is not conserved even though S_θ is invariant under general coordinate transformations. To see this, consider an infinitesimal diffeomorphism generated by an arbitrary vector field V^μ . The invariance of S_θ implies that, to first order in V^μ ,

$$S_\theta [g_{\mu\nu} + D_\mu V_\nu + D_\nu V_\mu, G + V^\mu D_\mu G, \Lambda + V^\mu D_\mu \Lambda] = S_\theta [g_{\mu\nu}, G, \Lambda]. \quad (2.18)$$

Using (2.7) this condition boils down to

$$D^\mu \theta_{\mu\nu} = \frac{1}{\sqrt{-g}} \left(\frac{\delta S_\theta}{\delta G} D_\nu G + \frac{\delta S_\theta}{\delta \Lambda} D_\nu \Lambda \right). \quad (2.19)$$

If G and Λ were conventional scalars satisfying equations of motion $\delta S_\theta/\delta G = 0 = \delta S_\theta/\delta \Lambda$ the RHS of (2.19) would vanish and $\theta_{\mu\nu}$ would be conserved. This is the standard argument leading to the conservation law (2.6). In the external field problem at hand the functional derivatives of S_θ with respect to G and Λ have no reason to vanish, however, and $\theta_{\mu\nu}$ is not conserved in general. (A trivial exception is the choice $S_\theta \equiv 0$ which is also investigated below.)

2.2 The consistency condition

The modified Einstein equation (2.10) is subject to a rather restrictive integrability condition. As a consequence of Bianchi's identities the divergence of its LHS vanishes identically, $D^\mu G_{\mu\nu} = 0$, and so the divergence of the RHS has to vanish, too:

$$D^\mu \Delta t_{\mu\nu} + D^\mu \vartheta_{\mu\nu} - D_\nu \Lambda + 8\pi (D_\mu G) T^\mu_\nu = 0. \quad (2.20)$$

Einstein's equation admits solutions only if this equation, henceforth referred to as the “consistency condition”, is satisfied. For $T_{\mu\nu}$ and $\vartheta_{\mu\nu}$ fixed, it can be regarded as a constraint on possible “backgrounds” $(G(x), \Lambda(x))$ which admit a consistent dynamics of the metric and the matter fields. Conversely, we could insist on a specific physically motivated background $(G(x), \Lambda(x))$. In this case eq. (2.20) is a condition on the tensor $\vartheta_{\mu\nu}$. From this point of view the consistency condition is highly welcome since, as we shall see, it can reduce the arbitrariness in the choice of $\theta_{\mu\nu}$ quite significantly.

By virtue of (2.4), the first term of the LHS of (2.20) is known explicitly,

$$\frac{1}{G} D^\mu G (\Delta t_{\mu\nu} - R_{\mu\nu}) + D^\mu \vartheta_{\mu\nu} - D_\nu \Lambda + 8\pi (D_\mu G) T^\mu_\nu = 0 \quad (2.21)$$

where (2.2) should be inserted for $\Delta t_{\mu\nu}$. For clarity we shall sometimes refer to (2.20) or (2.21) as the “consistency condition proper” or, for a reason which will become clear in a moment, as the “off-shell consistency condition”.

With G , Λ and $\vartheta_{\mu\nu}$ fixed, Einstein's equation and the consistency condition are two independent sets of equations for $g_{\mu\nu}$ which have to be solved simultaneously (together with the matter field equations). Therefore it is legitimate to insert one of the equations into the other, and to use the resulting new equation as the independent one. We take advantage of this freedom by eliminating the Ricci tensor in the consistency condition (2.21) by means of Einstein's equation in the form (2.13). The latter yields

$$R_{\mu\nu} - \Delta t_{\mu\nu} = Q_{\mu\nu} + \tilde{\vartheta}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \Delta t^\alpha_\alpha. \quad (2.22)$$

Inserting (2.22) with (2.3) into (2.21) we find

$$\begin{aligned} & \frac{3}{2} \frac{D_\nu G}{G^3} [G D^2 G - 2 (DG)^2] + D^\mu \vartheta_{\mu\nu} - \frac{D^\mu G}{G} \tilde{\vartheta}_{\mu\nu} \\ & + 4\pi T D_\nu G - \frac{1}{G} D_\nu (G\Lambda) = 0. \end{aligned} \quad (2.23)$$

Very often this alternative form of the consistency condition is more easily analyzed than the original one. Eqs. (2.20) and (2.23) are equivalent “on-shell”, i.e. only when $g_{\mu\nu}$

satisfies its field equation. We shall therefore refer to (2.23) as the “on-shell consistency condition”.

2.3 The Brans-Dicke type θ -tensor

Before continuing the general discussion let us look at an important special case. It is characterized by the absence on any matter, $T_{\mu\nu} = 0$, and a background $(G(x), \Lambda(x))$ with $G(x)\Lambda(x) = \text{const}$ which is realized in the fixed point regime, for instance.

In this special case the “on-shell” consistency condition (2.23) reduces to

$$D^\mu \vartheta_{\mu\nu} - \frac{D^\mu G}{G} \tilde{\vartheta}_{\mu\nu} + \frac{3}{2} \frac{D_\nu G}{G^3} [G D^2 G - 2 (DG)^2] = 0 \quad (2.24)$$

This is an equation for $\vartheta_{\mu\nu}$ as a function of G and its derivatives. In order to analyze it, it is convenient to introduce the field

$$\psi(x) \equiv -\ln [G(x) / \overline{G}] \quad (2.25)$$

so that $G = \overline{G} e^{-\psi}$ where \overline{G} is an arbitrary constant reference value. In terms of ψ , eq. (2.24) reads

$$D^\mu \vartheta_{\mu\nu} + \tilde{\vartheta}_{\mu\nu} D^\mu \psi + \frac{3}{2} D_\nu \psi [(D\psi)^2 + D^2 \psi] = 0. \quad (2.26)$$

In Appendix B we show that the *unique* tensor satisfying (2.26) identically in ψ and vanishing for $\psi = \text{const}$ is given by

$$\vartheta_{\mu\nu}^{\text{BD}} = -\frac{3}{2} \left[D_\mu \psi D_\nu \psi - \frac{1}{2} g_{\mu\nu} (D\psi)^2 \right] \quad (2.27)$$

$$= -\frac{3}{2 G^2} \left[D_\mu G D_\nu G - \frac{1}{2} g_{\mu\nu} (DG)^2 \right]. \quad (2.28)$$

This example nicely illustrates the power of the consistency condition: it has completely fixed the form of $\theta_{\mu\nu}^{\text{BD}} = \vartheta_{\mu\nu}^{\text{BD}} / 8\pi G$. However, this uniqueness property follows only if one demands that $\vartheta_{\mu\nu}$ satisfies (2.26) identically with respect to ψ , i.e. that the

consistency condition is satisfied for *all* background fields $\psi(x)$ or $G(x)$. Actually this is not necessary in our approach: it is sufficient that the consistency condition is satisfied by the specific background supplied by the RG equation. If the class of functions ψ is restricted to have special properties, additional solutions can exist. If ψ is assumed to solve $(D\psi)^2 + D^2\psi = 0$, say, $\vartheta_{\mu\nu} = 0$ is a solution of this kind.

Let us look more closely at the tensor

$$\theta_{\mu\nu}^{\text{BD}} = \left(-\frac{3}{2}\right) \frac{1}{8\pi G^3} \left[D_\mu G D_\nu G - \frac{1}{2} g_{\mu\nu} (DG)^2 \right]. \quad (2.29)$$

When reexpressed in terms of $\phi \equiv 1/G$, $\theta_{\mu\nu}^{\text{BD}}$ is seen to equal precisely the Brans-Dicke energy-momentum tensor $\mathcal{T}_{\mu\nu}^\omega$ of eq. (1.4) provided one sets $\omega = -3/2$ there.

Thus it might seem that, at least for this special case, the theory we have constructed is equivalent to standard Brans-Dicke theory whose coupled system of field equations for ϕ and $g_{\mu\nu}$ is consistent only if ϕ satisfies the Klein-Gordon equation

$$(3 + 2\omega) D^2\phi = 8\pi T. \quad (2.30)$$

For a generic value of ω this equation determines ϕ which cannot be treated as an externally prescribed field then. What comes to our rescue here is that $\omega = -3/2$ amounts to the singular limit of Brans-Dicke theory where (2.30) degenerates to the statement $T = 0$. This is no longer an equation of motion of ϕ but rather a constraint on the matter system. In the case at hand the trace of $T_{\mu\nu}$ vanishes trivially, and (2.30) is satisfied in the form $0 = 0$ for *any* function $D^2\phi(x)$. Therefore, as it should be, it does not fix the form of $\phi(x)$ in our case.

Even though we are not doing Brans-Dicke theory here, we shall refer to $\theta_{\mu\nu}^{\text{BD}}$ as the “ θ -tensor of Brans-Dicke type” because it has the same structure as $\mathcal{T}_{\mu\nu}^\omega$.

It is easily checked that, via eq. (2.7), $\theta_{\mu\nu}^{\text{BD}}$ can be obtained from the following action:

$$S_{\theta}^{\text{BD}}[g, G] = \frac{3}{32\pi} \int d^4x \sqrt{-g} \frac{D_{\mu}G D^{\mu}G}{G^3} \quad (2.31)$$

$$= \frac{3}{32\pi\overline{G}} \int d^4x \sqrt{-g} e^{\psi} D_{\mu}\psi D^{\mu}\psi. \quad (2.32)$$

Note that S_{θ}^{BD} does not depend on Λ .

As we argued already, $\theta_{\mu\nu}$ is not conserved in general. For the case of $\theta_{\mu\nu}^{\text{BD}}$ this can be demonstrated explicitly by taking the divergence of (2.29):

$$D^{\mu}\theta_{\mu\nu}^{\text{BD}} = \frac{3}{32\pi G^4} \left[3 (DG)^2 - 2 G D^2 G \right] D_{\nu}G. \quad (2.33)$$

For an alternative proof of this relation one can insert the action (2.31) into eq. (2.19). Note that the non-conservation of $\theta_{\mu\nu}^{\text{BD}}$ is not simply due to the x -dependence of G in (2.11); the tensor $\vartheta_{\mu\nu}^{\text{BD}}$, too, is not conserved:

$$D^{\mu}\vartheta_{\mu\nu}^{\text{BD}} = \frac{3}{2G^3} \left[(DG)^2 - G D^2 G \right] D_{\nu}G. \quad (2.34)$$

Next we resume the discussion of arbitrary $T_{\mu\nu}$'s and backgrounds.

2.4 Special classes of solutions

Up to this point, our discussion of RG improved action functionals has lead to a coupled system of effective field equations consisting of (1) Einstein's equation (2.10), (2) the equation of motion of the matter fields (2.17), and (3) the consistency condition (2.21).

Clearly it is very difficult in general to find solutions to this coupled system. Therefore, to start with, we discuss various classes of solutions which result from making certain simplifying assumptions. The specification of a class involves (i) a choice of $\theta_{\mu\nu}$, (ii) assumptions about the external fields $G(x)$, $\Lambda(x)$, and (iii) assumptions about the matter system. Some of the classes have quite remarkable properties which will be explored further in Subsection 2.5. In Section 3 on cosmology we impose the symmetry condition of homogeneity and isotropy which allows us to find explicit examples for all classes.

In this paper we investigate two choices of the θ -tensor: $\theta_{\mu\nu} \equiv 0$ and $\theta_{\mu\nu} = \theta_{\mu\nu}^{\text{BD}}$. The motivation for using the Brans-Dicke energy-momentum tensor is twofold: It enjoys the uniqueness property discussed in Subsection 2.3, and it allows for a comparison of standard Brans-Dicke theory with our “external field Brans-Dicke theory”.

To be specific, we shall deal with the following special cases:

Class I: This class is defined by the choice $\theta_{\mu\nu} \equiv 0$. Solutions of Class I satisfy

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \Delta t_{\mu\nu} + 8\pi G T_{\mu\nu} \quad (2.35a)$$

$$\frac{3}{2} \frac{D_\nu G}{G^2} [G D^2 G - 2 (DG)^2] + 4\pi G T D_\nu G - D_\nu (G\Lambda) = 0 \quad (2.35b)$$

along with $\delta S_M / \delta A = 0$.

Class II: This class is defined by an identically vanishing cosmological constant and a matter system whose energy-momentum tensor has a vanishing trace $T \equiv T_\mu{}^\mu$, at least “on-shell”:

$$\Lambda = 0 \quad \text{and} \quad T = 0. \quad (2.36a)$$

As a result, in Class II the on-shell consistency condition (2.23) reduces to eq. (2.24). In Subsection 2.3 we saw that the unique tensor solving this latter equation identically in ψ is $\vartheta_{\mu\nu}^{\text{BD}}$. Allowing for special properties of ψ there exist more general solutions, but as one of the defining properties of this specific class we include

$$\theta_{\mu\nu} = \theta_{\mu\nu}^{\text{BD}} \quad (2.36b)$$

into the definition of Class II. Thanks to (2.36a) and (2.36b) the on-shell consistency condition is satisfied by construction, and it remains to solve Einstein’s equation together with the matter field equation of motion.

Class III: This class is characterized by

$$\theta_{\mu\nu} = \theta_{\mu\nu}^{\text{BD}} \quad \text{and} \quad \Lambda \neq 0. \quad (2.37a)$$

The on-shell consistency condition (2.23) is not satisfied automatically, but it simplifies considerably. As a result of the choice $\theta_{\mu\nu} = \theta_{\mu\nu}^{\text{BD}}$, its first line vanishes identically and it remains to impose

$$4\pi G T \partial_\mu G = \partial_\mu (G\Lambda). \quad (2.37b)$$

It is convenient to distinguish two sub-classes of the Class III:

Class IIIa: This sub-class corresponds to the special case when

$$T = 0 \quad \text{and} \quad G\Lambda = \text{const.} \quad (2.38)$$

In this case the residual consistency condition (2.37b) is solved automatically, its LHS and RHS both being zero.

Class IIIb: In this sub-class the residual consistency condition is satisfied in a non-trivial way, i. e. not in the form “0 = 0”:

$$4\pi G T \partial_\mu G = \partial_\mu (G\Lambda) \neq 0. \quad (2.39)$$

It is quite intriguing that the RHS of the residual consistency condition (2.37b) vanishes precisely when the product $G\Lambda$ is constant, as it is the case in the fixed point regime, for instance. Assuming $\partial_\mu G \neq 0$, the LHS vanishes only if $T_{\mu\nu}$ is traceless, i. e. when the matter system is described by a quantum conformal field theory with vanishing trace anomaly, for instance. What eq. (2.37b) tells us is that when gravity is at a critical point (fixed point regime) so must be the matter fields. In this situation the combined gravity plus matter system can be regarded as a kind of scale invariant “critical phenomenon”. At least when we use $\theta_{\mu\nu}^{\text{BD}}$, only traceless matter can be coupled to gravity at its UV fixed point.

Henceforth we shall assume that the traceless $T_{\mu\nu}$ of the Classes II and IIIa originates from a Weyl-invariant matter action so that, for any function $\sigma(x)$,

$$S_{\text{M}} [e^{2\sigma} g_{\mu\nu}, e^{-2\Delta_A \sigma} A] = S_{\text{M}} [g_{\mu\nu}, A]. \quad (2.40)$$

(For simplicity we consider only a single matter field A with Weyl weight Δ_A .) The infinitesimal form of (2.40) reads

$$T_{\mu}^{\mu} = \frac{2\Delta_A}{\sqrt{-g}} \frac{\delta S_M}{\delta A} A \quad (2.41)$$

implying that $T_{\mu\nu}$ is indeed traceless when A is on-shell.

2.5 Solutions from a Weyl transformation

In this Subsection we discuss a very powerful tool for analyzing the Classes II and IIIa. As we shall see, solutions of these types can be obtained by simply Weyl-transforming solutions of the standard Einstein equations with constant G and Λ .

There exists an extensive literature on the use of Weyl transformations in gravitational theories and on the physical interpretation of the conformal frames they connect [31]. In the present paper, the Weyl transformations and the metrics $\gamma_{\mu\nu}$ they lead to (see below) should be regarded merely a technical tool for generating solutions of the modified field equations. They have no direct physical significance. It should also be mentioned that the standard discussion of Weyl rescalings applied to Brans-Dicke theory [31,32] does not apply in our case since it breaks down at the singular point $\omega = -3/2$ we are working at.

We start by picking two fixed reference values $\bar{\Lambda}$ and \bar{G} of the cosmological and Newton's constant, respectively, and we introduce the conventional Einstein-Hilbert actions

$$\bar{S}_{\text{EH}}[g] \equiv -\frac{1}{16\pi\bar{G}} \int d^4x \sqrt{-g} \left[-R(g) + 2\bar{\Lambda} \right], \quad (2.42)$$

$$\bar{S}_{\text{EH}}^0[g] \equiv \frac{1}{16\pi\bar{G}} \int d^4x \sqrt{-g} R(g). \quad (2.43)$$

For the corresponding functionals with the matter action included we write

$$\bar{S}_{\text{tot}}[g, A] \equiv \bar{S}_{\text{EH}}[g] + S_M[g, A], \quad (2.44)$$

$$\bar{S}_{\text{tot}}^0[g, A] \equiv \bar{S}_{\text{EH}}^0[g] + S_M[g, A]. \quad (2.45)$$

We assume that S_M is Weyl invariant.

Let us focus on **Class II** first. Here, by definition, $\Lambda = 0$ so that the total action reads

$$S_{\text{tot}}[g, G, A] = S_{\text{mEH}}[g, G, 0] + S_{\theta}^{\text{BD}}[g, G] + S_M[g, A] \quad (2.46)$$

Substituting $1/G = e^{\psi}/\bar{G}$ in eq. (1.19) and using (2.32) we find that

$$S_{\text{mEH}}[g, G, 0] + S_{\theta}^{\text{BD}}[g, G] = \frac{1}{16\pi\bar{G}} \int d^4x \sqrt{-g} e^{\psi} \left[R + \frac{3}{2} D_{\mu}\psi D^{\mu}\psi \right]. \quad (2.47)$$

Remarkably, the coefficient of the $(D\psi)^2$ -term in (2.47) is precisely such that this term can be absorbed into the R -term by means of a Weyl rescaling of the metric⁵ with $\sigma = \psi/2$:

$$S_{\text{mEH}}[g, G, 0] + S_{\theta}^{\text{BD}}[g, G] = \bar{S}_{\text{EH}}^0 \left[\frac{\bar{G}}{G} g_{\mu\nu} \right] \equiv \bar{S}_{\text{EH}}^0[\gamma_{\mu\nu}]. \quad (2.48)$$

The rescaled metric is $\gamma_{\mu\nu} = e^{\psi} g_{\mu\nu}$, or

$$\gamma_{\mu\nu}(x) = \frac{\bar{G}}{G(x)} g_{\mu\nu}(x). \quad (2.49a)$$

Thus the sum of S_{mEH} and S_{θ}^{BD} equals the *standard* Einstein-Hilbert action with a constant value of Newton's constant. The only place where the RG improvement manifests itself is the conformal factor $\bar{G}/G(x)$ which gets attached to the metric. The possibility of performing this Weyl transformation is directly related to the fact that we work in the singular limit of Brans-Dicke theory where $\omega = -3/2$ [33]. In standard Brans-Dicke theory there is no such possibility.

The matter action is Weyl invariant for any σ , and so in particular for $\sigma = \psi/2$. Hence it follows that $S_M[g_{\mu\nu}, A] = S_M[\gamma_{\mu\nu}, \mathcal{A}]$ with the rescaled matter field

$$\mathcal{A}(x) = \left[\frac{\bar{G}}{G(x)} \right]^{-\Delta_A} A(x). \quad (2.49b)$$

⁵Recall that the transformation $g'_{\mu\nu} = e^{2\sigma} g_{\mu\nu}$ gives rise to $\int d^4x \sqrt{-g'} = \int d^4x \sqrt{-g} e^{4\sigma}$ and $\int d^4x \sqrt{-g'} R(g') = \int d^4x \sqrt{-g} e^{2\sigma} \left[R(g) + 6 D_{\mu}\sigma D^{\mu}\sigma \right]$. In obtaining the second relation an integration by parts has been performed and the surface term has been dropped.

As a consequence, the total action (2.46) for a position-dependent G reduces to (2.45) for a constant G , up to a rescaling of the fields according to (2.49):

$$S_{\text{tot}}[g, G, A] = \overline{S}_{\text{tot}}^0[\gamma, \mathcal{A}]. \quad (2.50)$$

If a configuration $(g_{\mu\nu}, A)$ is a stationary point of the functional $S_{\text{tot}}[g, G, A]$, the related configuration $(\gamma_{\mu\nu}, \mathcal{A})$ is a stationary point of $\overline{S}_{\text{tot}}^0[\gamma, \mathcal{A}]$, i. e. it is a solution of the standard constant- G , $\Lambda = 0$ -Einstein equation

$$G_{\mu\nu}(\gamma) = 8\pi\overline{G} T_{\mu\nu}(\mathcal{A}, \gamma) \quad (2.51a)$$

and the matter equation

$$\delta\overline{S}_{\text{tot}}^0/\delta\mathcal{A} = 0. \quad (2.51b)$$

This observation provides us with a very efficient technique for obtaining the solutions of Class II: we take any solution $(\gamma_{\mu\nu}, \mathcal{A})$ of the much simpler system (2.51) and invert the Weyl transformation (2.49) in order to find $(g_{\mu\nu}, A)$:

$$g_{\mu\nu}(x) = \frac{G(x)}{\overline{G}} \gamma_{\mu\nu}(x) \quad (2.52a)$$

$$A(x) = \left[\frac{G(x)}{\overline{G}} \right]^{-\Delta_A} \mathcal{A}(x). \quad (2.52b)$$

For $G(x)$ regular, the Weyl transformation is invertible and *all* solutions can be found in this manner.

Until now we assumed that the cosmological constant vanishes. If we allow for an arbitrary function $\Lambda(x)$, the Λ -term in S_{mEH} ,

$$\frac{1}{16\pi} \int d^4x \sqrt{-g} \left\{ -2 \frac{\Lambda(x)}{G(x)} \right\},$$

makes it impossible to use a Weyl transformation in order to convert $G(x)$ to \overline{G} everywhere. However, there is one exception to this rule, namely when $\Lambda(x)$ is proportional

to $1/G(x)$, i.e. when the product $G(x)\Lambda(x)$ is constant. This situation corresponds precisely to the solutions of **Class IIIa** to which we turn next. We write the constant product in the form

$$G(x)\Lambda(x) = \overline{G}\overline{\Lambda}, \quad (2.53)$$

whence $\Lambda(x)/G(x) = e^{2\psi}\overline{\Lambda}/\overline{G}$. As a result, the Λ -term, too, can be brought to its constant- G , constant- Λ form by the Weyl rescaling (2.49a):

$$\frac{1}{16\pi} \int d^4x \sqrt{-g} \left\{ -2 \frac{\Lambda(x)}{G(x)} \right\} = \frac{1}{16\pi\overline{G}} \int d^4x \sqrt{-\gamma} \left\{ -2 \overline{\Lambda} \right\}. \quad (2.54)$$

Again we observe that the case $G\Lambda = \text{const}$ which is special from the RG point of view because it is realized in the fixed point regime enjoys very special properties also with respect to the gravitational actions and field equations. Combining (2.50) with (2.54) we see that for the Class IIIa the total action can be reduced to the one with the x -independent G and Λ :

$$S_{\text{tot}}[g, G, \Lambda, A] = \overline{S}_{\text{tot}}[\gamma, \mathcal{A}]. \quad (2.55)$$

The stationary points $(g_{\mu\nu}, A)$ of $S_{\text{tot}}[g, G, \Lambda, A]$ are related to the stationary points $(\gamma_{\mu\nu}, \mathcal{A})$ of $\overline{S}_{\text{tot}}[\gamma, \mathcal{A}]$ by the same transformations as above, eqs. (2.49). The latter are solutions to the conventional constant- G , constant- Λ field equations

$$G_{\mu\nu}(\gamma) = -\overline{\Lambda}\gamma_{\mu\nu} + 8\pi\overline{G}T_{\mu\nu}(\mathcal{A}, \gamma) \quad (2.56a)$$

$$\delta\overline{S}_{\text{tot}}/\delta\mathcal{A} = 0. \quad (2.56b)$$

Thus we conclude that the solutions of the Class IIIa can be obtained by Weyl-transforming the solutions of the system (2.56) according to eqs. (2.52). There is no corresponding simplification in Class IIIb.

Clearly the Weyl transformation $\gamma_{\mu\nu} = e^\psi g_{\mu\nu}$ can always be performed, whatever is $\theta_{\mu\nu}$, $T_{\mu\nu}$, G , and Λ . It takes us from the “ $g_{\mu\nu}$ -frame” to the “ $\gamma_{\mu\nu}$ -frame” in which $\gamma_{\mu\nu}$ is

the independent variable and where the modified Einstein equation reads

$$G_{\mu\nu}(\gamma) = \frac{3}{2}(1 - \varepsilon) \left[D_\mu \psi D_\nu \psi - \frac{1}{2} \gamma_{\mu\nu} \gamma^{\alpha\beta} D_\alpha \psi D_\beta \psi \right] + \left[-\Lambda(x) \gamma_{\mu\nu} + 8\pi \overline{G} T_{\mu\nu}(A, e^{-\psi} \gamma) \right] e^{-\psi} \quad (2.57)$$

In writing down (2.57) we adopted the choice $\theta_{\mu\nu} = \varepsilon \theta_{\mu\nu}^{\text{BD}}$ where ε is an arbitrary real constant, with $\varepsilon = 0$ and $\varepsilon = 1$ being particularly interesting cases, of course. Only in the classes II and IIIa ψ drops out from the RHS of (2.57). In general this Einstein equation is reminiscent of ordinary gravity plus a massless Klein-Gordon field. While occasionally this analogy is helpful for generating solutions, it is quite deceptive from the physical point of view because physical lengths are still measured by the metric $g_{\mu\nu}$ and not by $\gamma_{\mu\nu}$. For instance, when one applies this formalism to black holes [34] one would like the distance function $d(r)$ appearing in the cutoff identification (1.12) to be computed from the actual metric of the improved spacetime, i.e. from $g_{\mu\nu}$. If $g_{\mu\nu}$ is to be regarded as the product $e^\psi \gamma_{\mu\nu}$ the situation becomes very involved because then the cutoff identification we insert into $G(k)$, besides $\gamma_{\mu\nu}$, becomes explicitly dependent on $\psi(x)$, i.e. $G(x)$, itself. In order to avoid complications of this kind we shall mostly employ the physical “ $g_{\mu\nu}$ -frame”.

In Brans-Dicke jargon the $g_{\mu\nu}$ - and the $\gamma_{\mu\nu}$ -frame are called the Jordan and the Einstein frame, respectively. There is a longstanding debate in the literature about the issue of which conformal frame is the physical one. Ref. [31] contains a detailed discussion of this problem, and it is argued there that only the Einstein frame can be physical. A key role in establishing this argument is played by the positivity of the energy and by the existence and the stability of a ground state in the Einstein frame. If this discussion applied also to the theory of RG improved actions it would give preference to $\gamma_{\mu\nu}$, rather than $g_{\mu\nu}$, as the physical metric. However, we emphasize that none of the arguments put forward in [31] is applicable to the “external field Brans-Dicke theory”. In particular we stress that there can be no doubt that the “Jordan” metric $g_{\mu\nu}$ is the physical one in our case. This is an immediate consequence of the effective field theory approach we

are employing: $g_{\mu\nu}$, and not $\gamma_{\mu\nu}$, is the average of the microscopic metric, the variable of integration in the path-integral over all metrics [6]. By including matter terms into the flow equation it is obvious that it is $g_{\mu\nu}$ which enters the conservation law $D_\mu T^{\mu\nu} = 0$ and determines the geodesics of test particles. The standard discussion does not apply here for a variety of reasons: (i) As we mentioned already, $\omega = -3/2$ is a highly singular point. (ii) In the standard case, the scalar field $G(x)$ exists in an unambiguous, i.e. “process independent” way. Not so in our case: A priori G is neither position nor time but rather *scale* dependent, and only under very special conditions, and not everywhere in spacetime, the k -dependence can be translated to a x -dependence. This invalidates, for instance, standard discussions of the equivalence principle and the “universality” of the gravitational interaction.⁶ (iii) The positivity of the action and the properties of the ground state must be checked on the basis of the exact $\Gamma \equiv \Gamma_{k=0}$ [4]. Since only an approximate form of Γ_k , $k > 0$ is known, we are clearly not yet in a position to address questions about the vacuum of Quantum Einstein Gravity.

3 Cosmological evolution equations

In the remaining sections of this paper we apply the approach developed in Section 2 to the cosmology of homogeneous and isotropic Universes.

Because of this symmetry requirement, the metric can be brought to the Robertson-

⁶According to a conservative interpretation of “universality” the present theory does imply “nonuniversal” effects, in the sense that the coupling strength $G(k)$ depends on features such as the size of the bodies involved in the process considered, or on their relative momentum, say (cf. the discussion in the Introduction). However, in accordance with the terminology in particle physics, we do not consider it appropriate to call a certain interaction “nonuniversal” just because the corresponding coupling displays RG running.

Walker form

$$ds^2 = -dt^2 + a^2(t) d\Omega_K^2 \quad (3.1)$$

where

$$d\Omega_K^2 \equiv \frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (3.2)$$

is the line element of a maximally symmetric 3-space of positive ($K = +1$), negative ($K = -1$), or vanishing ($K = 0$) curvature, respectively.

The scalar functions $G(x)$ and $\Lambda(x)$ are assumed to respect all symmetries. This implies that in the (t, r, θ, φ) coordinate system which we shall use throughout they may depend on the cosmological time only: $G(x) = G(t)$ and $\Lambda(x) = \Lambda(t)$.

In the same coordinate system, the energy-momentum tensor is of the form

$$T_\mu^\nu = \text{diag}[-\rho, p, p, p] \quad (3.3)$$

which is consistent with the symmetries provided the density ρ and pressure p depend on t only. Here, rather than in terms of a field A , we describe the matter sector in a hydrodynamical language in terms of a perfect fluid. Because of the strong symmetry constraint, the only information which is needed about the matter system is its equation of state $p = p(\rho)$. For the time being we leave it unspecified. As always, T_μ^ν is assumed to be conserved, $D_\nu T_\mu^\nu = 0$, which implies

$$\dot{\rho} + 3H(\rho + p) = 0, \quad (3.4)$$

where $H(t) \equiv \dot{a}(t)/a(t)$ denotes the Hubble parameter.

Upon inserting the Robertson-Walker metric into (2.2) we obtain

$$\Delta T_\mu^\nu \equiv \frac{1}{8\pi G} \Delta t_\mu^\nu = \text{diag}[-\Delta\rho, \Delta p, \Delta p, \Delta p] \quad (3.5)$$

where

$$\Delta\rho = \frac{-\Delta t_0^0}{8\pi G} = \frac{3}{8\pi G} \left(\frac{\dot{G}}{G} \right) H \quad (3.6a)$$

and

$$\Delta p = \frac{\Delta t_i^i}{8\pi G} = \frac{1}{8\pi G} \left[2 \left(\frac{\dot{G}}{G} \right)^2 - 2 \left(\frac{\dot{G}}{G} \right) H - \frac{\ddot{G}}{G} \right] \quad (3.6b)$$

is the density and pressure, respectively, which is contributed by the extra terms in the Einstein equation due to the t -dependence of G . (Eq. (3.6b) holds for any $i = 1, 2, 3$; this index is not summed over.) Likewise, every choice of $\theta_{\mu\nu}$ leads to only two independent functions, ρ_θ and p_θ , in a Robertson-Walker background:

$$\theta_\mu^\nu = \text{diag} [-\rho_\theta, p_\theta, p_\theta, p_\theta]. \quad (3.7)$$

Plugging the metric (3.1) into Einstein's equation (2.9) we obtain two independent equations⁷: its 00-component

$$H^2 + \frac{K}{a^2} = \frac{8\pi}{3} G (\rho + \rho_\Lambda + \Delta\rho + \rho_\theta). \quad (3.8a)$$

and the ii -components which are identical for all values of the spatial index $i = 1, 2, 3$:

$$H^2 + 2 \left(\frac{\ddot{a}}{a} \right) + \frac{K}{a^2} = -8\pi G (p + p_\Lambda + \Delta p + p_\theta). \quad (3.8b)$$

In writing down eqs. (3.8) we also introduced the vacuum energy density and pressure, respectively, which are due to the cosmological constant:

$$\rho_\Lambda \equiv \frac{\Lambda(t)}{8\pi G(t)}, \quad p_\Lambda \equiv -\frac{\Lambda(t)}{8\pi G(t)}. \quad (3.9)$$

⁷With our conventions and notations the non-zero components of the Einstein tensor are

$$G_0^0 = -3 \left[H^2 + \frac{K}{a^2} \right] \quad \text{and} \quad G_i^j = - \left[H^2 + 2 \left(\frac{\ddot{a}}{a} \right) + \frac{K}{a^2} \right] \delta_i^j.$$

We shall find it convenient to define a “critical” energy density in the same way as in standard cosmology,

$$\rho_{\text{crit}} \equiv \frac{3}{8\pi G(t)} H^2(t), \quad (3.10)$$

and to refer the matter and vacuum density to ρ_{crit} :

$$\Omega_{\text{M}} \equiv \frac{\rho}{\rho_{\text{crit}}}, \quad \Omega_{\Lambda} \equiv \frac{\rho_{\Lambda}}{\rho_{\text{crit}}}. \quad (3.11a)$$

The analogous relative energy densities due to $\Delta T_{\mu\nu}$ and $\theta_{\mu\nu}$ are

$$\Delta\Omega \equiv \frac{\Delta\rho}{\rho_{\text{crit}}}, \quad \Omega_{\theta} \equiv \frac{\rho_{\theta}}{\rho_{\text{crit}}}. \quad (3.11b)$$

In the language of the Ω 's, the modified Friedmann equation, i. e. the 00-component (3.8a), reads

$$K = \dot{a}^2 [\Omega_{\text{M}} + \Omega_{\Lambda} + \Delta\Omega + \Omega_{\theta} - 1]. \quad (3.12)$$

We observe that in order to obtain a spatially flat, expanding Universe all four densities, Ω_{M} , Ω_{Λ} , $\Delta\Omega$, and Ω_{θ} , must add up to unity.

As for the consistency condition, its original off-shell form (2.20) yields:

$$\dot{G} (\rho + \rho_{\Lambda} + \Delta\rho + \rho_{\theta}) + G \frac{d}{dt} (\rho_{\Lambda} + \Delta\rho + \rho_{\theta}) + 3 G H [\Delta\rho + \Delta p + \rho_{\theta} + p_{\theta}] = 0. \quad (3.13)$$

At this point the system of coupled cosmological evolution equations consists of the 00-component of Einstein's equation (3.8a), its ii -component (3.8b), the off-shell consistency condition (3.13), and the equation of state $p = p(\rho)$. As in standard cosmology, it is possible to replace the ii -component with the continuity equation $D_{\mu}T^{\mu\nu} = 0$, i. e. with (3.4), as an independent relation. In Appendix C we prove that every cosmology with $\dot{a} \neq 0$ which satisfies the 00-component, the off-shell consistency condition, and the continuity equation automatically also satisfies the ii -component of Einstein's equation. This statement holds true for any choice of the θ -tensor.

As we are now going to discard the ii -component as an independent equation of motion it is no longer guaranteed that the consistency condition proper, the “off-shell” condition (2.20), is fully equivalent to the “on-shell” condition (2.23). We shall therefore employ the off-shell condition in the following. (The on-shell condition is nevertheless useful as any solution has to satisfy it, of course.)

Let us fix some “background” by picking two functions $G(t)$ and $\Lambda(t)$. We would then like to determine $a(t)$, $\rho(t)$, and $p(t)$ from the coupled system of cosmological evolution equations consisting of

1. the 00-component of Einstein’s equation
 2. the off-shell consistency condition
 3. the continuity equation
 4. the equation of state
- (3.14)

Obviously we are trying to determine *three* functions from *four* independent equations. As a result, an arbitrarily chosen background (G, Λ) and tensor $\theta_{\mu\nu}$, generically, will not allow for any solution of the system (3.14). Being over-determined, the system (3.14) puts restrictions on (G, Λ) and the θ -tensor. As we mentioned already in Section 2, this is very welcome because $\theta_{\mu\nu}$ is not completely determined by general principles. In the ideal case when the RG equations yield a certain trajectory $(G(k), \Lambda(k))$ and a physically plausible cutoff identification $k = k(t)$ turns it into a background $(G(t), \Lambda(t))$ for which (3.14) is indeed soluble, there exists a rather non-trivial conspiracy of the RG- and field-equations. Being comparatively rare one is inclined to ascribe a particularly high degree of physical relevance to such “precious” solutions.

Next we discuss the special choices $\theta_{\mu\nu} = 0$ and $\theta_{\mu\nu} = \theta_{\mu\nu}^{\text{BD}}$ in turn.

(a) The choice $\theta_{\mu\nu} = 0$

For a vanishing θ -tensor the 00- and ii -component of Einstein's equation read, respectively,

$$H^2 + \frac{K}{a^2} = \frac{1}{3} \Lambda + \frac{8\pi}{3} G \rho + \left(\frac{\dot{G}}{G} \right) H \quad (3.15a)$$

$$H^2 + 2 \left(\frac{\ddot{a}}{a} \right) + \frac{K}{a^2} = \Lambda - 8\pi G p - 2 \left(\frac{\dot{G}}{G} \right)^2 + \frac{\ddot{G}}{G} + 2 \left(\frac{\dot{G}}{G} \right) H \quad (3.15b)$$

The off-shell consistency condition is obtained by starting from (2.21) and inserting Δt_0^0 from (3.6a), $\theta_{\mu\nu} = 0$, and $R_0^0 = 3\ddot{a}/a$ which is true for any Robertson-Walker metric. The only non-trivial condition follows from $\nu = 0$:

$$\dot{\Lambda} + 8\pi \dot{G} \rho + 3 \left(\frac{\dot{G}}{G} \right)^2 H + 3 \left(\frac{\dot{G}}{G} \right) \left(\frac{\ddot{a}}{a} \right) = 0. \quad (3.16)$$

As a check, it can be verified explicitly that (3.15a), (3.16), and the continuity equation imply (3.15b).

(b) The choice $\theta_{\mu\nu} = \theta_{\mu\nu}^{\text{BD}}$

Specializing eq. (2.29) we find for the energy and pressure contribution due to $\theta_{\mu\nu}$:

$$\rho_\theta = \frac{-\vartheta_0^0}{8\pi G} = -\frac{3}{32\pi G} \left(\frac{\dot{G}}{G} \right)^2, \quad (3.17)$$

$$p_\theta = \frac{\vartheta_i^i}{8\pi G} = -\frac{3}{32\pi G} \left(\frac{\dot{G}}{G} \right)^2 \quad (i \text{ not summed}).$$

The 00- and ii -components of Einstein's equation are correspondingly

$$H^2 + \frac{K}{a^2} = \frac{1}{3} \Lambda + \frac{8\pi}{3} G \rho + \left(\frac{\dot{G}}{G} \right) H - \frac{1}{4} \left(\frac{\dot{G}}{G} \right)^2, \quad (3.18a)$$

$$H^2 + 2 \left(\frac{\ddot{a}}{a} \right) + \frac{K}{a^2} = \Lambda - 8\pi G p - \frac{5}{4} \left(\frac{\dot{G}}{G} \right)^2 + \frac{\ddot{G}}{G} + 2 \left(\frac{\dot{G}}{G} \right) H. \quad (3.18b)$$

For the Brans-Dicke choice of the θ -tensor the *on-shell* form of the consistency condition, eq. (2.23), reduces to the much simpler residual condition (2.37b). In cosmology, with $T = 3p - \rho$, the latter boils down to

$$4\pi (3p - \rho) \dot{G}G = \frac{d}{dt}(G\Lambda). \quad (3.19)$$

This condition is equivalent to the original (“off-shell”) version of the consistency condition, eq. (2.20) or (2.21), only when *all* field equations are used. Rather than (3.19) the system (3.14) must include the *off-shell* consistency condition (2.21) which assumes the form

$$\dot{\Lambda} + 8\pi \dot{G} \rho - \frac{3}{2} \left(\frac{\dot{G}}{G} \right)^2 H + 3 \left(\frac{\dot{G}}{G} \right) \left(\frac{\ddot{a}}{a} \right) + \frac{3}{2} \left(\frac{\dot{G}}{G} \right)^3 - \frac{3}{2} \left(\frac{\ddot{G}\dot{G}}{G^2} \right) = 0. \quad (3.20)$$

In deriving (3.20) from (2.21) we took $D^\mu \vartheta_{\mu\nu}^{\text{BD}}$ from eq. (2.34), and we exploited that G is a spatially constant scalar on which the D’Alembertian D^2 acts according to $-D^2 G = \ddot{G} + 3H\dot{G}$.

Again it can be checked by a somewhat tedious calculation that (3.18a) together with the off-shell consistency condition (3.20) and the continuity equation implies (3.18b), provided $a(t)$ is non-constant.

Up to this point of the discussion the equation of state was kept completely arbitrary. For the practical calculations in the following sections we adopt the linear ansatz

$$p(\rho) = w \rho \quad (3.21)$$

where w is a constant. As a result, the continuity equation

$$\dot{\rho} + 3(1 + w) H \rho = 0 \quad (3.22)$$

is easily solved for ρ as a function of a :

$$\rho(t) = \frac{\mathcal{M}}{8\pi [a(t)]^{3+3w}}. \quad (3.23)$$

Here \mathcal{M} is a constant of integration with mass dimension $1 - 3w$, defined in the same way as in ref. [18]. With (3.21) the trace of the energy-momentum tensor is

$$T = 3p - \rho = (3w - 1) \rho. \quad (3.24)$$

It vanishes if $w = 1/3$ (“radiation dominance”) or if $\mathcal{M} = 0$ (vacuum solutions).

4 Solutions of Class I

In this section we discuss various cosmological solutions of Class I. By definition, $\theta_{\mu\nu} = 0$ in this class. For the equation of state $p = w \rho$ the relevant evolution equations are the modified Friedmann equation (3.15a) and the off-shell consistency condition (3.16) with (3.21) and (3.23) inserted:

$$H^2 + Ka^{-2} = \frac{1}{3}\Lambda + \frac{1}{3}\mathcal{M}Ga^{-3-3w} + H \left(\frac{\dot{G}}{G} \right), \quad (4.1a)$$

$$\dot{\Lambda} + \mathcal{M}\dot{G}a^{-3-3w} + 3H \left(\frac{\dot{G}}{G} \right)^2 + 3 \left(\frac{\dot{G}}{G} \right) \left(\frac{\ddot{a}}{a} \right) = 0. \quad (4.1b)$$

As these are two independent equations for one function, $a(t)$, it is to be expected that the system is not soluble for every $G(t)$, $\Lambda(t)$. Nevertheless, we shall find analytic solutions in the fixed point regime of the RG flow, as well as for more general power laws $G \propto t^n$, $\Lambda \propto 1/t^2$. Let us discuss these examples in turn.

4.1 Fixed point solution with $K = 0$

In this subsection we present a power law solution to the eqs. (4.1) with $G(t)$ and $\Lambda(t)$ given by (1.21). This is the time dependence which is expected to occur when the underlying RG trajectory is close to a fixed point. The discussion is valid for a UV- and IR-fixed point alike. In the former case we describe the very early Universe ($k \rightarrow \infty$, $t \rightarrow 0$), while

the latter refers to asymptotically late times ($k \rightarrow 0$, $t \rightarrow \infty$). To start with let us look at spatially flat Universes, $K = 0$.

Inserting a power law ansatz $a \propto t^\alpha$ into (4.1) we find that indeed *both* equations can be satisfied by a scale factor of this type provided the constant $\tilde{\lambda}_*$ assumes the value

$$\lambda_* \xi^2 \equiv \tilde{\lambda}_* = \frac{2 (5 - 3w)}{3 (1 + w)^2}. \quad (4.2)$$

Since λ_* is fixed once we have picked a specific RG trajectory, eq. (4.2) is to be interpreted as an equation for ξ :

$$\xi^2 = \frac{2 (5 - 3w)}{3 (1 + w)^2} \frac{1}{\lambda_*}. \quad (4.3)$$

The condition (4.2) and, as a consequence, the possibility of actually *computing* the factor of proportionality relating k to $1/t$, is a direct consequence of the fact that the system of evolution equations is over-determined; it admits solutions with the “external field” $\Lambda(t) = \tilde{\lambda}_*/t^2$ only for one specific value of $\tilde{\lambda}_*$.

Henceforth, with an eye towards the UV fixed point found in [11–14, 23] and the IR fixed point postulated in [19] we assume that $g_* > 0$ and $\lambda_* > 0$. As a result, solutions exist only in the range $w < 5/3$ because otherwise ξ^2 becomes negative. In the rest of this subsection we make the more restrictive assumption that $-1 < w < 5/3$.

The $K = 0$ fixed point cosmology thus obtained is explicitly given by

$$a(t) = \left[-\frac{(1+w)^3}{4(5-3w)} \mathcal{M} g_* \lambda_* \right]^{1/(3+3w)} t^{4/(3+3w)}, \quad (4.4a)$$

$$\rho(t) = -\frac{(5-3w)}{2\pi(1+w)^3} \frac{1}{g_* \lambda_*} t^{-4}, \quad (4.4b)$$

$$G(t) = \frac{3(1+w)^2}{2(5-3w)} g_* \lambda_* t^2, \quad (4.4c)$$

$$\Lambda(t) = \frac{2(5-3w)}{3(1+w)^2} \frac{1}{g_* \lambda_*} t^{-2}. \quad (4.4d)$$

Along with the scale factor we also wrote down the energy density according to (3.21) as well as $G(t)$ and $\Lambda(t)$ with ξ eliminated everywhere by using (4.3). We observe that after eliminating ξ the observables (4.4) depend on the fixed point coordinates g_* and λ_* only via their product. This is a rather nontrivial and encouraging result because the values of g_* and λ_* are scheme dependent, hence unphysical, while their product $g_*\lambda_*$ is not [7, 11, 14]. Observables should depend on this product only.

At first sight the solution (4.4) has a certain similarity with the cosmology (A.2) obtained by improving Einstein's equation rather than the action. However, there is a crucial and, as we shall see, rather symptomatic difference: Since $g_*\lambda_* > 0$ and $w < 5/3$, the density (4.4b) is *negative*. Hence the conserved quantity $\mathcal{M} \equiv 8\pi \rho a^{3+3w}$ is negative, too, and this is in fact what is needed to make the scale factor (4.4a) real. Clearly a negative density prevents us from interpreting ρ as the energy density due to ordinary (baryonic) matter.

In order to understand this point better we note that for the cosmology (4.4) the “critical” energy density (3.10), which is positive by definition, is given by

$$\rho_{\text{crit}} = \frac{4(5-3w)}{9\pi(1+w)^4} \frac{1}{g_*\lambda_*} t^{-4}. \quad (4.5)$$

All other densities of interest, ρ , ρ_Λ , $\Delta\rho$, and ρ_θ , are proportional to ρ_{crit} , the constants of proportionality being

$$\begin{aligned} \Omega_M &= -\frac{9}{8}(1+w) < 0, & \Omega_\Lambda &= \frac{1}{8}(5-3w) > 0, \\ \Delta\Omega &= \frac{3}{2}(1+w) > 0, & \Omega_\theta &= 0. \end{aligned} \quad (4.6)$$

According to (3.12), every $K=0$ cosmology in Class I obeys

$$\Omega_M + \Omega_\Lambda + \Delta\Omega = 1, \quad (4.7)$$

and clearly the Ω 's of (4.6) satisfy (4.7). It is important to note that the new term in (4.7), $\Delta\Omega$, which is absent both in standard cosmology and in the approach of improving

equations, makes always a strictly positive contribution to the LHS of this equation. Since Ω_Λ cannot become negative for $\lambda_* > 0$, the additional positive contribution in (4.7) must be compensated by a smaller positive, or even negative value of Ω_M , as compared to the usual situation where $\Omega_M + \Omega_\Lambda = 1$. This is precisely what we found: Ω_M turned out negative because it has to counteract a too strongly positive $\Delta\Omega$ -contribution.

The physical interpretation of this phenomenon is as follows. Contrary to the approach of improving equations, in the present approach of improved actions the energy and momentum carried by the field $G(x)$ acts as a source of spacetime curvature. This happens in two different ways: via the tensor $\Delta T_{\mu\nu}$ which follows straightforwardly from the variational principle, and via $\theta_{\mu\nu}$. In a situation where both approaches are applicable, and reliable, they should lead to similar results, at least at a qualitative level. This implies that, roughly speaking, $\Delta T_{\mu\nu}$ and $\theta_{\mu\nu}$ cancel one another to some extent. Typically, $\Delta T_{\mu\nu}$ supplies a positive energy density, as in the example above, and $\theta_{\mu\nu}$ a negative one.

The prime example is the Brans-Dicke energy-momentum tensor (2.27) which (apart from a factor of $3/2$) is the *negative* of the energy-momentum tensor of an ordinary scalar ψ , and correspondingly S_θ^{BD} of (2.31) contains a kinetic term of the “wrong” sign. This becomes explicit in cosmology where eqs. (3.6a) and (3.10) lead to

$$\Delta\Omega = \frac{1}{H} \left(\frac{\dot{G}}{G} \right) \quad (4.8)$$

which is always positive if $\dot{G} > 0$ and $\dot{a} > 0$, whereas eq. (3.17) yields

$$\Omega_\theta^{\text{BD}} = -\frac{1}{4H^2} \left(\frac{\dot{G}}{G} \right)^2 \quad (4.9)$$

which is strictly negative. Similar remarks apply to the pressure.

Above we tried the choice $\theta_{\mu\nu} = 0$, and we found that the corresponding solution of the new approach, eqs. (4.4), looks quite different from its counterpart in the old approach, eqs. (A.2). The reason is that, as there is no θ -tensor, the positive contributions from

$\Delta T_{\mu\nu}$ must be compensated by negative contributions which are now forced into $T_{\mu\nu}$ rather than $\theta_{\mu\nu}$. As a result, we cannot interpret $T_{\mu\nu}$ as the energy-momentum of the ordinary baryonic matter alone. $T_{\mu\nu}$ is “contaminated” by stress-energy contributions stemming from the fields $G(x)$ and $\Lambda(x)$. Therefore the energy density pertaining to $T_{\mu\nu}$ is to be interpreted as a sum $\rho = \rho_{\text{mat}} + \rho_{G,\Lambda}$ where ρ_{mat} is due to the ordinary matter and $\rho_{G,\Lambda}$ to the energy carried by the fields $G(x)$ and $\Lambda(x)$.

In general we have no tool for disentangling ρ_{mat} from $\rho_{G,\Lambda}$; the evolution equations determine their sum only. However, in the light of the discussion above it is a plausible assumption that $\rho_{G,\Lambda}$ is approximately the negative of $\Delta\rho$. With $\rho_{G,\Lambda} = -\Delta\rho$ we have $\rho = \rho_{\text{mat}} - \Delta\rho$ so that, under this hypothesis, it is the sum $\rho + \Delta\rho$ which should be identified with the ordinary matter energy density.

In order to show that this is actually true we define the energy-momentum tensor

$$\widehat{T}_\mu{}^\nu \equiv T_\mu{}^\nu + \Delta T_\mu{}^\nu \equiv \text{diag}[-\widehat{\rho}, \widehat{p}, \widehat{p}, \widehat{p}] \quad (4.10)$$

with the entries $\widehat{\rho} \equiv \rho + \Delta\rho$ and $\widehat{p} \equiv p + \Delta p$. Eq. (3.6a) yields for the cosmology (4.4)

$$\Delta\rho = \frac{2(5-3w)}{3\pi(1+w)^3} \frac{1}{g_*\lambda_*} t^{-4} \quad (4.11)$$

and adding the ρ of (4.4b) leads to

$$\widehat{\rho}(t) = -\frac{1}{3}\rho(t). \quad (4.12)$$

This density is indeed positive and can be identified with ρ_{mat} therefore. It will be convenient to define a conserved quantity $\widehat{\mathcal{M}}$ in terms of $\widehat{\rho}$ in the same way as \mathcal{M} is defined in terms of ρ , $\widehat{\mathcal{M}} \equiv 8\pi\widehat{\rho}a^{3+3w}$, satisfying

$$\widehat{\mathcal{M}} = -\frac{1}{3}\mathcal{M} > 0. \quad (4.13)$$

It is instructive to rewrite the solution (4.4) in terms of $\widehat{\rho}$ and $\widehat{\mathcal{M}}$:

$$a(t) = \left[\frac{3(1+w)^3}{4(5-3w)} \widehat{\mathcal{M}} g_* \lambda_* \right]^{1/(3+3w)} t^{4/(3+3w)}, \quad (4.14a)$$

$$\widehat{\rho}(t) = \frac{(5-3w)}{6\pi(1+w)^3} \frac{1}{g_* \lambda_*} t^{-4}, \quad (4.14b)$$

$$G(t) = \frac{3(1+w)^2}{2(5-3w)} g_* \lambda_* t^2, \quad (4.14c)$$

$$\Lambda(t) = \frac{2(5-3w)}{3(1+w)^2} \frac{1}{g_* \lambda_*} t^{-2}. \quad (4.14d)$$

$$\widehat{p}(t) = \frac{(5-3w)}{18\pi(1+w)^3} \frac{1}{g_* \lambda_*} t^{-4}. \quad (4.14e)$$

In the above list we included the pressure $\widehat{p} = w\rho + \Delta p$ with

$$\Delta p = \frac{(5-3w)(1+9w)}{18\pi(1+w)^3} \frac{1}{g_* \lambda_*} t^{-4} \quad (4.15)$$

as obtained from (3.6b) with (4.4).

We note in passing that while p and ρ satisfy a simple equation of state, $p(\rho) = w\rho$, the relationship between Δp and $\Delta\rho$, or \widehat{p} and $\widehat{\rho}$, is very complicated in general, see eqs. (3.6). However, for the solution at hand we find a remarkably simple and intriguing “equation of state” for \widehat{p} and $\widehat{\rho}$:

$$\widehat{p} = \frac{1}{3} \widehat{\rho} \quad \Longleftrightarrow \quad \widehat{T}_\mu{}^\mu = 0. \quad (4.16)$$

We emphasize that this relation holds true for *any* value of w . The modified energy-momentum tensor is always traceless, while the original $T_{\mu\nu}$ is only for $w = 1/3$.

Let us now compare the equations (4.14), obtained by improving the action, to eqs. (A.2) which resulted from improving the field equations. Looking at the case $w = 1/3$ first, we find that the two cosmologies are *completely identical* if one identifies the ρ and \mathcal{M} of (A.2) with the $\widehat{\rho}$ and $\widehat{\mathcal{M}}$ in (4.14). This confirms our hypothesis that in the Class I-solutions it is the sum $\rho + \Delta\rho$ which is to be identified with ρ_{mat} , while in the framework

of improved equations it is ρ itself. Even the result for the pressure is the same in both cases since, by (4.16), we have $\hat{p} = \hat{\rho}/3$ for any w , and this equation of state accidentally coincides with $p = w\rho$ if $w = 1/3$.

Also for other values of w in the interval $(-1, 5/3)$ the two cosmologies are qualitatively similar; the time dependencies of all quantities of interest are the same, only the prefactors of the various powers of t differ slightly. Let us denote the functions (A.2) where \mathcal{M} is replaced with $\widehat{\mathcal{M}}$ by a_{ieq} , ρ_{ieq} , G_{ieq} , and Λ_{ieq} , respectively, with “ieq” standing for “improved equation”. Comparing them to their analogs in (4.14) we find the time-independent ratios

$$\frac{a(t)}{a_{\text{ieq}}(t)} = \left[\frac{3}{16} (1+w) (5-3w) \right]^{-1/(3+3w)} \quad (4.17a)$$

$$\frac{\hat{\rho}(t)}{\rho_{\text{ieq}}(t)} = \frac{3}{16} (1+w) (5-3w) \quad (4.17b)$$

$$\frac{G(t)}{G_{\text{ieq}}(t)} = \frac{4}{5-3w} \quad (4.17c)$$

$$\frac{\Lambda(t)}{\Lambda_{\text{ieq}}(t)} = \frac{1}{4} (5-3w). \quad (4.17d)$$

For $w = 1/3$ all ratios are exactly equal to 1, and for w arbitrary but not too close to the boundaries of the interval $(-1, 5/3)$ they are still rather close to unity. For $w = 0$, say, one has 0.98, 0.94, 0.80, and 1.25, respectively.

Thus we may conclude that with the reinterpretation of ρ_{mat} as $\rho + \Delta\rho$ the two approaches, improving equations and improving actions, lead to very similar results, the ratios (4.17) being a measure of their quantitative precision.

The only minor difference concerns the pressure. The corresponding ratio is

$$\frac{\hat{p}(t)}{p_{\text{ieq}}(t)} = \frac{(1+w) (5-3w)}{16w} \quad (w \neq 0) \quad (4.18)$$

It can become large when w is close to zero. In fact, for $w = 0$ we have $p_{\text{ieq}} = 0$, but

$$\hat{p}(t) = \Delta p(t) = \frac{5}{18\pi} \frac{1}{g_* \lambda_*} t^{-4} \quad (4.19)$$

is nonzero.

4.2 Fixed point solutions with $K = \pm 1$

In the case of a spatially curved Universe, too, it is possible to find solutions of Class I with $G(t)$ and $\Lambda(t)$ evolving according to the fixed point law (1.21), albeit only for the equation of state with

$$w = 1/3. \quad (4.20)$$

From eqs. (4.1) with $K = +1$ or -1 we obtain the following cosmologies with a linearly growing scale factor:

$$a(t) = \left[\frac{1}{3} \left(-K + 2 \sqrt{1 - \frac{1}{3} \mathcal{M} g_* \lambda_*} \right) \right]^{1/2} t \quad (4.21a)$$

$$\rho(t) = \frac{9}{8\pi} \frac{\mathcal{M}}{\left(-K + 2 \sqrt{1 - \frac{1}{3} \mathcal{M} g_* \lambda_*} \right)^2} t^{-4} \quad (4.21b)$$

$$G(t) = \frac{1}{3} \frac{\left(-K + 2 \sqrt{1 - \frac{1}{3} \mathcal{M} g_* \lambda_*} \right)}{\left(K + \sqrt{1 - \frac{1}{3} \mathcal{M} g_* \lambda_*} \right)} g_* \lambda_* t^2 \quad (4.21c)$$

$$\Lambda(t) = 3 \frac{\left(K + \sqrt{1 - \frac{1}{3} \mathcal{M} g_* \lambda_*} \right)}{\left(-K + 2 \sqrt{1 - \frac{1}{3} \mathcal{M} g_* \lambda_*} \right)} \frac{1}{g_* \lambda_*} t^{-2}. \quad (4.21d)$$

As in the case $K = 0$, the evolution equations fix the value of $\tilde{\lambda}_*$. In writing down (4.21) we used this information in order to express ξ in terms of λ_* everywhere. They are related by

$$\xi^2 = \frac{3}{2} \left[1 + \frac{K}{A^2} \right] \frac{1}{\lambda_*} \quad (4.22)$$

where $A^2 \equiv \frac{1}{3} \left[-K + 2 \sqrt{1 - \mathcal{M} g_* \lambda_* / 3} \right]$. To make sense, ξ must be real, and this condition leads to a constraint on the “matter” contents of the Universe as parameterized

by the constant \mathcal{M} . Assuming, as always, $g_* > 0$ and $\lambda_* > 0$, it reads

$$\begin{aligned}\mathcal{M} &< \frac{9}{4g_*\lambda_*} && \text{for } K = +1 \\ \mathcal{M} &< 0 && \text{for } K = -1.\end{aligned}\tag{4.23}$$

If (4.23) is satisfied, the scale factor as well as G and Λ are real and positive in (4.21). For $K = -1$ the situation is similar as in the previous subsection: the energy density ρ is negative for all allowed values of \mathcal{M} . A new phenomenon is encountered in the spherical case $K = +1$. Here there exists a “window” of \mathcal{M} -values between zero and $9/(4g_*\lambda_*)$ for which the density ρ is positive.

Again, the various energy densities are all proportional to the critical energy density,

$$\rho_{\text{crit}} = \frac{9}{8\pi} \frac{\left(K + \sqrt{1 - \frac{1}{3}\mathcal{M}g_*\lambda_*}\right)}{\left(-K + 2\sqrt{1 - \frac{1}{3}\mathcal{M}g_*\lambda_*}\right)} \frac{1}{g_*\lambda_*} t^{-4},\tag{4.24}$$

but the corresponding Ω ’s depend on \mathcal{M} now:

$$\begin{aligned}\Omega_{\text{M}} &= \frac{\mathcal{M}g_*\lambda_*}{\left(K + \sqrt{1 - \frac{1}{3}\mathcal{M}g_*\lambda_*}\right) \left(-K + 2\sqrt{1 - \frac{1}{3}\mathcal{M}g_*\lambda_*}\right)}, \\ \Omega_{\Lambda} &= \frac{\left(K + \sqrt{1 - \frac{1}{3}\mathcal{M}g_*\lambda_*}\right)}{\left(-K + 2\sqrt{1 - \frac{1}{3}\mathcal{M}g_*\lambda_*}\right)},\end{aligned}\tag{4.25}$$

$$\Delta\Omega = 2, \quad \Omega_{\theta} = 0.$$

Note that $\Delta\Omega$ has the same positive value as for $K = 0$ with $w = 1/3$.

A Vacuum Solution: Picking $K = +1$, the cosmology (4.21) has a well-behaved limit⁸ $\mathcal{M} \rightarrow 0$. It describes a linearly expanding Universe with spherical time slices which does not contain any real matter; the expansion is driven by $\Delta T_{\mu\nu}$ and the cosmological

⁸With $K = -1$, Newton’s constant (4.21c) diverges in the limit $\mathcal{M} \rightarrow 0$. Also the $K = 0$ cosmology (4.4) becomes singular in this limit, its scale factor $a(t)$ vanishes identically.

constant alone. Apart from the universal product $g_*\lambda_*$, this cosmology does not involve any free parameter:

$$a(t) = \frac{1}{\sqrt{3}}t, \quad (4.26a)$$

$$\rho(t) = 0, \quad (4.26b)$$

$$G(t) = \frac{1}{6}g_*\lambda_*t^2, \quad (4.26c)$$

$$\Lambda(t) = 6t^{-2}. \quad (4.26d)$$

It is easily checked explicitly that this limiting case satisfies all relevant evolution equations. The parameter ξ is fixed by $\xi^2 = 6/\lambda_*$, and the densities are $\Omega_M = 0$, $\Omega_\Lambda = 2$, $\Delta\Omega = 2$, $\Omega_\theta = 0$.

This vacuum solution owes its existence to the tensor $\Delta T_{\mu\nu}$; improving the field equation rather than the action one finds no analogous solution. This is a further indication that in the vacuum sector the two approaches can yield similar results only when an appropriate θ -tensor is included.

4.3 General power laws ($K = 0$)

Being over-determined, the system of equations (4.1) restricts the allowed backgrounds $(G(t), \Lambda(t))$. This feature reduces ambiguities related to the non-universal properties of the RG flow and to the mathematical modeling of the physical cutoff mechanism by the identification $k = k(x)$. In the fixed point regime the restrictions were rather mild, only the parameter $\tilde{\lambda}_*$ got fixed. We shall now investigate backgrounds of the type

$$G(t) = C t^n, \quad \Lambda(t) = D t^{-m} \quad (4.27)$$

where $C > 0$, D , n , and m are a priori arbitrary real constants. Here the restrictions on allowed backgrounds are seen much more clearly. The only power law solutions admitted by (4.1), with $K = 0$, $w \geq -1$, and $\mathcal{M} \neq 0$, are the following three families.

1. First family ($n \neq -2$, $w > -1$)

The solutions belonging to this family are labeled by an arbitrary real exponent n , different from -2 , and by the parameters C and \mathcal{M} . Imposing solubility of the system, m and D are uniquely determined:

$$a(t) = \left[\frac{3(1+w)^2}{4-3n(1+w)-n^2(4+3w)} \mathcal{M} C \right]^{1/(3+3w)} t^{(n+2)/(3+3w)}, \quad (4.28a)$$

$$\rho(t) = \frac{1}{24\pi} \frac{[4-3n(1+w)-n^2(4+3w)]}{(1+w)^2} \frac{1}{C} t^{-(n+2)}, \quad (4.28b)$$

$$G(t) = C t^n, \quad (4.28c)$$

$$\Lambda(t) = \frac{n}{3} \frac{(2n+1-3w)}{(1+w)^2} t^{-2}. \quad (4.28d)$$

For the special choice $n = +2$, $C = \frac{3}{2} (1+w)^2 g_* \lambda_* / (5-3w)$, eqs. (4.28) reproduce the fixed point solution (4.4). A new feature of (4.28) is that if n lies in a narrow interval $[n_-, n_+]$ there exist solutions with positive energy density ($\mathcal{M} > 0$). These solutions are more the exception than the rule, however; for all $n < n_-$ and $n > n_+$ the cosmology (4.28) exists only if $\mathcal{M} < 0$. (Here we assume $C > 0$ so that G is positive.) The limits of the interval depend on the equation of state:

$$n_{\pm} = -\frac{3}{2} \left(\frac{1+w}{4+3w} \right) \pm \left[\frac{4}{(4+3w)} + \frac{9}{4} \left(\frac{1+w}{4+3w} \right)^2 \right]^{1/2} \quad (4.29)$$

For dust and radiation, say,

$$\begin{aligned} n_- &\approx -1.44, \quad n_+ \approx 0.69 & (w = 0) \\ n_- &\approx -1.38, \quad n_+ \approx 0.58 & (w = 1/3) \end{aligned} \quad (4.30)$$

so that the window for positive energy solutions is indeed comparatively small.

2. Second family ($n = -2$, $w > -1$)

There exists an exceptional cosmology where both Λ and G decay $\propto t^{-2}$. Assuming, as always, that $C > 0$ it exists only for negative \mathcal{M} and ρ :

$$a(t) = \left[-\frac{1}{2} (1+w) \mathcal{M} C \right]^{1/(3+3w)} = \text{const}, \quad (4.31a)$$

$$\rho(t) = -\frac{1}{4\pi} \frac{1}{(1+w)} \frac{1}{C} = \text{const}, \quad (4.31b)$$

$$G(t) = C t^{-2}, \quad (4.31c)$$

$$\Lambda(t) = \frac{2}{(1+w)} t^{-2}. \quad (4.31d)$$

This cosmology is quite exotic in that G and Λ depend on time but the Universe does not expand.⁹ Even though the scale factor is constant, this Universe has an initial singularity (“big bang”) at which G and Λ diverge.

3. Third family ($w = -1$)

For the equation of state with $w = -1$ there exists another exotic cosmology with a time independent density, and with a , G , and Λ increasing proportional to \sqrt{t} :

$$a(t) = A t^{1/2}, \quad (4.32a)$$

$$\rho(t) = \frac{\mathcal{M}}{8\pi}, \quad (4.32b)$$

$$G(t) = C t^{1/2}, \quad (4.32c)$$

$$\Lambda(t) = -\mathcal{M} C t^{1/2}. \quad (4.32d)$$

The normalization of the scale factor, A , is completely arbitrary.

⁹Since $\dot{a} = 0$, the ii -component of Einstein’s equation must be checked explicitly; it is indeed found to be satisfied by (4.31).

5 Solutions of Class II

According to Subsection 2.4, the Class II is defined by $\theta_{\mu\nu} = \theta_{\mu\nu}^{\text{BD}}$, $\Lambda = 0$, and $T = 0$ which translates into $w = 1/3$ or $\mathcal{M} = 0$ in the present setting. The relevant cosmological evolution equations are the modified Friedmann equation (3.18a) and the off-shell consistency condition (3.20) with $\Lambda \equiv 0$, $w = 1/3$, and $\rho = \mathcal{M}/(8\pi a^4)$ inserted:

$$H^2 + \frac{K}{a^2} = \frac{\mathcal{M}}{3} G a^{-4} + \left(\frac{\dot{G}}{G}\right) H - \frac{1}{4} \left(\frac{\dot{G}}{G}\right)^2, \quad (5.1a)$$

$$\mathcal{M} \dot{G} a^{-4} - \frac{3}{2} \left(\frac{\dot{G}}{G}\right)^2 H + 3 \left(\frac{\dot{G}}{G}\right) \left(\frac{\ddot{a}}{a}\right) + \frac{3}{2} \left(\frac{\dot{G}}{G}\right)^3 - \frac{3}{2} \left(\frac{\ddot{G}\dot{G}}{G^2}\right) = 0 \quad (5.1b)$$

Vacuum solutions are obtained from (5.1) with $\mathcal{M} = 0$.

5.1 Generating solutions via Weyl transformations

According to the discussion in Subsection 2.5 it should be possible to obtain the solutions to the system (5.1) by Weyl-transforming the solutions of the much simpler system (2.51) in which Newton's constant is really constant. There arises the following problem, however. Assume we have solved the simpler system and obtained a line element $ds_\gamma^2 \equiv \gamma_{\mu\nu} dx^\mu dx^\nu$ which has the standard Robertson-Walker form (3.1). According to (2.52a) the line element we are actually after, $ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu$, obtains as

$$ds^2 = \frac{G(t)}{\overline{G}} ds_\gamma^2, \quad (5.2)$$

and this is not of the Robertson-Walker form. As we shall discuss next, this defect can be repaired by a reparametrization of the time coordinate.

Let us write ds_γ^2 in the style of (3.1),

$$ds_\gamma^2 = -d\tau^2 + b^2(\tau) d\Omega_K^2 \quad (5.3)$$

with the cosmological time τ and the scale factor $b(\tau)$. Inserting (5.3) into the constant- G Einstein equation (2.51a) leads to the classical equations

$$H_b^2 + \frac{K}{b^2} = \frac{8\pi\overline{G}}{3} \check{\rho} \quad (5.4a)$$

$$H_b^2 + \frac{2}{b} \frac{d^2 b}{d\tau^2} + \frac{K}{b^2} = -\frac{8\pi\overline{G}}{3} \check{\rho} \quad (5.4b)$$

with $H_b \equiv b^{-1} (db/d\tau)$. We wrote $\check{\rho}$ and $\check{\rho}/3$ for the density and pressure corresponding to the energy-momentum tensor in the transformed frame, $T_{\mu\nu}(\mathcal{A}, \gamma)$. The substitute for the \mathcal{A} -equation of motion is the continuity equation

$$\frac{d}{d\tau} \check{\rho}(\tau) + 4 H_b(\tau) \check{\rho}(\tau) = 0 \quad (5.5)$$

which integrates to $\check{\rho} = \mathcal{M}/(8\pi b^4)$ whence the 00- and ii -components of the constant- G equation become

$$H_b^2 + \frac{K}{b^2} = \frac{1}{3} \mathcal{M} \overline{G} b^{-4} \quad (5.6a)$$

$$H_b^2 + \frac{2}{b} \frac{d^2 b}{d\tau^2} + \frac{K}{b^2} = -\frac{1}{3} \mathcal{M} \overline{G} b^{-4}. \quad (5.6b)$$

We know that (5.4a) with (5.5) implies (5.4b), and that (5.6a) implies (5.6b) if $b \neq \text{const.}$ We adopt (5.6a) as the (only) independent b -equation.

Let us pick a solution $b(\tau)$. Then, in the $(\tau, r, \theta, \varphi)$ -coordinate system, the metric $g_{\mu\nu}$ is represented by

$$\begin{aligned} ds^2 &= [G/\overline{G}] \left(-d\tau^2 + b^2(\tau) d\Omega_K^2 \right) \\ &= -\left(\sqrt{G/\overline{G}} d\tau \right)^2 + \left(\sqrt{G/\overline{G}} b(\tau) \right)^2 d\Omega_K^2 \end{aligned} \quad (5.7)$$

where G is considered a function of τ a priori. Now we introduce a new time coordinate $t = t(\tau)$ such that (5.7) assumes the standard form $ds^2 = -dt^2 + a^2(t) d\Omega_K^2$. Obviously we need that $dt = \sqrt{G/\overline{G}} d\tau$, and since we would like to prescribe G in the final t - rather

than the original τ -coordinate system this condition provides us with the derivative of $\tau = \tau(t)$,

$$\frac{d}{dt} \tau(t) = \sqrt{\overline{G}/G(t)}, \quad (5.8)$$

which can be integrated immediately,

$$\tau(t) = \int_{t_1}^t dt' \sqrt{\overline{G}/G(t')}, \quad (5.9a)$$

with a constant t_1 . The final result for the scale factor a is obtained by a combined Weyl and general coordinate transformation:

$$a(t) = \sqrt{G(t)/\overline{G}} b(\tau(t)). \quad (5.9b)$$

The equations (5.9a) and (5.9b) express the “magic” of the cosmological Class II-solutions. Rather than dealing with the complicated-looking system (5.1) directly it is sufficient to solve the standard Friedmann equation of the radiation dominated Universe, eq. (5.6a). Because of this hidden simplicity, the system (5.1) is not over-determined, as was its analog for $\theta_{\mu\nu} = 0$. It is soluble for arbitrary prescribed functions $G(t)$.

When one inserts (5.9) into (5.1a) and (5.1b) it is impressive to see explicitly that both of those rather complicated equations are satisfied identically if $b(\tau)$ solves (5.6a). The calculation makes essential use of the following relations between the first and second derivatives of $a(t)$ and $b(\tau)$:

$$H = \sqrt{\overline{G}/G(t)} H_b + \frac{1}{2} \left(\frac{\dot{G}}{\overline{G}} \right) \quad (5.10)$$

$$\frac{\ddot{a}}{a} = [\overline{G}/G(t)] \left(\frac{1}{b} \frac{d^2 b}{d\tau^2} \right) - \frac{1}{2} \left(\frac{\dot{G}}{\overline{G}} \right)^2 + \frac{1}{2} \left(\frac{\ddot{G}}{\overline{G}} \right) + \frac{1}{2} H \left(\frac{\dot{G}}{\overline{G}} \right) \quad (5.11)$$

(As usual, the dot denotes the derivative with respect to t .)

In checking the a -equations it becomes obvious that the constant of integration \mathcal{M} must have the same value in the b - and the a -system of equations. In fact, only then the respective energy densities are related by

$$\rho(t) = [G(t) / \overline{G}]^{-2} \check{\rho}(\tau(t)) \quad (5.12)$$

so that ρ transforms according to (2.52b) with the expected Weyl weight $\Delta_\rho = 2$.

5.2 Vacuum spacetimes

The simplest situation with $T = 0$ is the absence of any matter so that $T_{\mu\nu} = 0$ and $\rho, \check{\rho}, \mathcal{M} = 0$. The relevant solutions to the b -equations (5.4) are Minkowski space,

$$b(\tau) = b_0 = \text{const}, \quad K = 0, \quad (5.13)$$

and the Milne Universe which is merely an unconventional coordinatization of flat space-time with hyperbolic time slices:

$$b(\tau) = \tau, \quad K = -1. \quad (5.14)$$

According to the discussion of the previous subsection, Minkowski space gives rise to an improved cosmology with

$$a(t) = b_0 \sqrt{G(t) / \overline{G}}, \quad K = 0, \quad (5.15)$$

and $\rho \equiv 0, \Lambda = 0$. Here $G(t)$ can be any function of time. The corresponding Hubble constant is $H = \dot{G} / (2G)$, whence, with (4.8) and (4.9),

$$\Omega_M = 0, \quad \Omega_\Lambda = 0, \quad \Delta\Omega = 2, \quad \Omega_\theta = -1. \quad (5.16)$$

We observe that the density contribution of $\theta_{\mu\nu}^{\text{BD}}$ indeed counteracts the one from $\Delta T_{\mu\nu}$, without completely canceling it though. Interestingly enough, also in this class of cosmologies the distinguished time dependence $G \propto t^2$ leads to a linear expansion of the Universe, $a \propto t$.

In an analogous fashion the Milne Universe generates the following vacuum solution, with zero cosmological constant and hyperbolic 3-space:

$$a(t) = \int_{t_1}^t dt' \sqrt{G(t)/G(t')}, \quad K = -1. \quad (5.17)$$

For $G \propto t^2$, say, this scale factor behaves as $a \propto t \ln t$.

There are no further vacuum solutions of Class II beyond (5.15) and (5.17), in particular there are none for $K = +1$. This can be seen directly from the Friedmann equation (5.1a) which, provided $\mathcal{M} = 0$, can be cast into the remarkable form

$$\left[\dot{a} - \frac{1}{2} \left(\frac{\dot{G}}{G} \right) a \right]^2 = -K. \quad (5.18)$$

Obviously (5.18) cannot be satisfied for $K = +1$, while for $K = 0$ and $K = -1$, respectively, its most general solutions (with $a > 0$) are given by (5.15) and (5.17).

5.3 A radiation dominated Universe

The simplest example of a non-vacuum spacetime with $T = 0$ is induced by the familiar spatially flat radiation dominated Universe with zero cosmological constant:

$$b(\tau) = \left[\frac{4}{3} \mathcal{M} \overline{G} \right]^{1/4} \tau^{1/2}, \quad K = 0. \quad (5.19)$$

Its Weyl-transform describes a $K = 0$ -cosmology with scale factor

$$a(t) = \left[\frac{4}{3} \mathcal{M} \right]^{1/4} \left(G(t) \int_{t_1}^t \frac{dt'}{\sqrt{G(t')}} \right)^{1/2}, \quad (5.20)$$

density $\rho = \mathcal{M}/(8\pi a^4)$, pressure $p = \rho/3$, and $\Lambda(t) = 0$.

The important point to be noted here is that this solution exists if, and only if, $\mathcal{M} > 0$. Thus the Universe is filled with matter of positive energy density ρ . Therefore, contrary to the $\theta_{\mu\nu} = 0$ -case discussed in the previous section, we may now interpret ρ as

the energy density of ordinary baryonic matter. In accordance with our earlier discussion, the Brans-Dicke θ -tensor included here compensates for certain contributions of $\Delta T_{\mu\nu}$, which then no longer “contaminate” ρ and p with contributions which are actually due to the position-dependence of G .

6 Solutions of Class III

In Class III, $\theta_{\mu\nu} = \theta_{\mu\nu}^{\text{BD}}$ and $\Lambda \neq 0$. The corresponding evolution equations, the 00-Einstein equation and the off-shell consistency condition, read, respectively,

$$H^2 + \frac{K}{a^2} = \frac{1}{3} \Lambda + \frac{1}{3} \mathcal{M} G a^{-3-3w} + H \left(\frac{\dot{G}}{G} \right) - \frac{1}{4} \left(\frac{\dot{G}}{G} \right)^2, \quad (6.1a)$$

$$\dot{\Lambda} + \mathcal{M} \dot{G} a^{-3-3w} - \frac{3}{2} H \left(\frac{\dot{G}}{G} \right)^2 + 3 \left(\frac{\ddot{a}}{a} \right) \left(\frac{\dot{G}}{G} \right) + \frac{3}{2} \left(\frac{\dot{G}}{G} \right)^3 - \frac{3}{2} \left(\frac{\ddot{G} \dot{G}}{G^2} \right) = 0 \quad (6.1b)$$

Generically this system of equations cannot be solved by the Weyl-transformation technique. An exception are the solutions in the sub-class IIIa to which we turn first.

6.1 The Class IIIa

According to our earlier definition, the Class IIIa is characterized by $G\Lambda = \text{const}$ and $T = 0$, i. e. by

$$\Lambda(t) = \frac{\overline{G} \overline{\Lambda}}{G(t)}, \quad \text{and} \quad w = \frac{1}{3} \quad \text{or} \quad \mathcal{M} = 0. \quad (6.2)$$

Very much as in the Class II, solutions of this type can be obtained by Weyl-transforming solutions of the corresponding constant- G , constant- Λ Einstein equation (2.56a), this time with $\overline{\Lambda} \neq 0$ though. As a result, the $b(\tau)$ -system reads

$$H_b^2 + \frac{K}{b^2} = \frac{1}{3} \overline{\Lambda} + \frac{1}{3} \mathcal{M} \overline{G} b^{-4}, \quad (6.3a)$$

$$H_b^2 + \frac{2}{b} \frac{d^2 b}{d\tau^2} + \frac{K}{b^2} = \overline{\Lambda} - \frac{1}{3} \mathcal{M} \overline{G} b^{-4}. \quad (6.3b)$$

Here we put $w = 1/3$; to get the corresponding vacuum equations one sets $\mathcal{M} = 0$. Every solution $b(\tau)$ of (6.3) induces a solution $a(t)$ of (6.1) with (6.2). As before, it is given by (5.9b) with (5.9a). The construction works for any given function $G(t) \propto 1/\Lambda(t)$.

The Class IIIa includes the fixed point regime where $G = \tilde{g}_* t^2$, $\Lambda = \tilde{\lambda}_* t^{-2}$, and $\overline{G}\overline{\Lambda} = \tilde{g}_*\tilde{\lambda}_*$. For this time dependence, the relationship between t and τ is given by

$$\tau(t) = \sqrt{\overline{G}/\tilde{g}_*} \ln(t/t_1). \quad (6.4)$$

Next we look at some instructive examples.

6.1.1 Vacuum solutions from (anti) de Sitter space

For $\mathcal{M} = 0$, the b -system (6.3) has the following well-known solutions which describe (a part of) de Sitter space ($\overline{\Lambda} > 0$) or anti-de Sitter space ($\overline{\Lambda} < 0$), respectively:

$$b(\tau) = b_0 \exp\left(\pm \sqrt{\overline{\Lambda}/3} \tau\right) \quad (K = 0, \overline{\Lambda} > 0) \quad (6.5a)$$

$$b(\tau) = \sqrt{3/\overline{\Lambda}} \cosh\left(\sqrt{\overline{\Lambda}/3} \tau\right) \quad (K = +1, \overline{\Lambda} > 0) \quad (6.5b)$$

$$b(\tau) = \sqrt{3/\overline{\Lambda}} \sinh\left(\sqrt{\overline{\Lambda}/3} \tau\right) \quad (K = -1, \overline{\Lambda} > 0) \quad (6.5c)$$

$$b(\tau) = \sqrt{-3/\overline{\Lambda}} \cos\left(\sqrt{-\overline{\Lambda}/3} \tau\right) \quad (K = -1, \overline{\Lambda} < 0). \quad (6.5d)$$

Restricting ourselves to the fixed point regime, these scale factors imply the following solutions to the original system (6.1) with $\mathcal{M} = 0$:

$$a(t) = A_{\pm} t^{1 \pm \nu} \quad (K = 0, \tilde{\lambda}_* > 0) \quad (6.6a)$$

$$a(t) = \frac{t}{2\nu} \left[\left(\frac{t}{t_1}\right)^{\nu} + \left(\frac{t_1}{t}\right)^{\nu} \right] \quad (K = +1, \tilde{\lambda}_* > 0) \quad (6.6b)$$

$$a(t) = \frac{t}{2\nu} \left[\left(\frac{t}{t_1}\right)^{\nu} - \left(\frac{t_1}{t}\right)^{\nu} \right] \quad (K = -1, \tilde{\lambda}_* > 0) \quad (6.6c)$$

$$a(t) = \frac{t}{\nu} \cos[\nu \ln(t/t_1)] \quad (K = -1, \tilde{\lambda}_* < 0). \quad (6.6d)$$

Here $\nu \equiv \sqrt{|\tilde{\lambda}_*|/3} > 0$ and $A_{\pm} \equiv b_0 t_1^{\mp\nu} \sqrt{\tilde{g}_*/\overline{G}}$.

6.1.2 Solutions from the Einstein static Universe

Another well-known solution of the b -equations with $\overline{\Lambda} > 0$, $K = +1$ is the Einstein static Universe, here filled with radiation rather than dust:

$$b = \left(\frac{3}{2} \frac{1}{\overline{\Lambda}}\right)^{1/2}, \quad \check{\rho} = \frac{1}{8\pi} \frac{\overline{\Lambda}}{\overline{G}}. \quad (6.7)$$

For arbitrary $G(t)$ it generates the solution

$$a(t) = \sqrt{\frac{3G(t)}{2\overline{G}\overline{\Lambda}}}, \quad \rho(t) = \frac{\overline{G}\overline{\Lambda}}{8\pi G(t)^2}, \quad (K = +1). \quad (6.8)$$

In the fixed point regime it describes a scale-free, linearly expanding Universe of positive spatial curvature:

$$a(t) = \left(\frac{3}{2} \frac{1}{\tilde{\lambda}_*}\right)^{1/2} t, \quad \rho(t) = \frac{1}{8\pi} \frac{\tilde{\lambda}_*}{\tilde{g}_*} t^{-4}. \quad (6.9)$$

This cosmology is quite similar to the attractor solution found in [18] by improving the Einstein equations. The only difference is that ξ does not get fixed in the present approach.

6.1.3 Another radiation Universe

As a last example, we consider the standard spatially flat, radiation dominated Universe with a positive cosmological constant:

$$b(\tau) = \left[\frac{\mathcal{M}\overline{G}}{2\overline{\Lambda}} \left\{ \cosh \left(4 \sqrt{\overline{\Lambda}/3} \tau \right) - 1 \right\} \right]^{1/4}. \quad (6.10)$$

In the fixed point regime with $\tilde{\lambda}_* > 0$, it generates a $K = 0$, $w = 1/3$ cosmology with the scale factor

$$a(t) = \left(\frac{\mathcal{M}\tilde{g}_*}{4\tilde{\lambda}_*} \right)^{1/4} t \left[\left(\frac{t}{t_1} \right)^{4\nu} + \left(\frac{t_1}{t} \right)^{4\nu} - 2 \right]^{1/4}. \quad (6.11)$$

For $\nu < 1$ this cosmology has no initial singularity.

6.2 The Class IIIb

In Class IIIb, which constitutes the generic case and is much larger than Class IIIa, solutions cannot be found by the Weyl-transformation technique. One has to work directly with the system of equations (6.1). In general it is over-determined and cannot be solved for arbitrary backgrounds $(G(t), \Lambda(t))$. To find some illustrative examples of cosmologies in Class IIIb we confine ourselves to the power law backgrounds (4.27) which we employed in Class I already. It turns out that (6.1) possesses the following three families of power law solutions with $K = 0$, $w \geq -1$, and $\mathcal{M} \neq 0$.

1. First family $(n \neq \pm 2, n \neq 4/(1+3w), w > -1, w \neq -1/3)$

This family of solutions is the Class IIIb-counterpart to the Class I-solutions in the “First family” of Subsection 4.3. Its members are labeled by the free constants \mathcal{M} and $C > 0$, while n and D are fixed by the requirement of solubility:

$$a(t) = \left[\frac{6(1+w)^2}{(n-2)(n+3nw-4)} \mathcal{M} C \right]^{1/(3+3w)} t^{(n+2)/(3+3w)}, \quad (6.12a)$$

$$\rho(t) = \frac{1}{48\pi} \frac{(n-2)(n+3nw-4)}{(1+w)^2} \frac{1}{C} t^{-(n+2)}, \quad (6.12b)$$

$$G(t) = C t^n, \quad (6.12c)$$

$$\Lambda(t) = \frac{n(3w-1)(n+3nw-4)}{12(1+w)^2} t^{-2}. \quad (6.12d)$$

For $n = 0$ this result reduces to the classical Friedmann cosmology with $K = 0$ and $\Lambda = 0$. In the general case, the allowed values of the exponent n are subject to certain restrictions which depend on the sign of \mathcal{M} . The solution exists and $a(t)$ is real if n , w , and \mathcal{M} are such that

$$(n-2)(n+3nw-4)/\mathcal{M} > 0. \quad (6.13)$$

For $w = 1/3$, say¹⁰, this requirement is met by any real $n \neq \pm 2$ if $\mathcal{M} > 0$, and it cannot be satisfied at all if $\mathcal{M} < 0$. Thus, for an almost arbitrary exponent n , we always obtain a solution with a positive matter energy density ρ .

Taking $w = 0$ as a second example, we see that for all exponents $n < 2$ and $n > 4$ we again have $\mathcal{M} > 0$ and positive energy density therefore. Only for $2 < n < 4$ we need a negative \mathcal{M} and ρ .

It is instructive to compare the Class IIIb-cosmology (6.12) to its Class I-analog (4.28) which was computed for the same time dependence of G . We observe that the various functions contain the same powers of t but different prefactors, and those different prefactors lead to different conditions for the existence of positive and negative energy density solutions, respectively. In Class I, solutions with $\mathcal{M} > 0$ exist only in a finite interval of n -values; the $\mathcal{M} < 0$ -solutions are more abundant and are realized in two infinite bands of exponents. In Class IIIb the situation is exactly the other way around: The $\mathcal{M} > 0$ -solutions are the more abundant ones and obtain for an infinite range of n -values. Solutions with $\mathcal{M} < 0$ exist at most within a finite n -interval.

These findings provide a further confirmation of the general picture we developed earlier. In Class III, contrary to Class I, a θ -tensor is included which can absorb the energy and momentum carried by $G(x)$ and $\Lambda(x)$. As a consequence, the solutions do not need to “squeeze” those contributions into $T_{\mu\nu}$ which then has a chance of describing ordinary matter with positive energy density and pressure.

¹⁰Strictly speaking, for this particular equation of state the cosmology (6.12) belongs to Class II rather than IIIb because it fulfills both defining conditions, $w = 1/3$ and $\Lambda = 0$. It is more convenient to consider it as a special member of the above family, however. It can also be obtained as a Weyl-transform by using (5.20) with $G(t) = C t^n$, $n \neq 2$, and $t_1 = 0$ (for $n < 2$) or $t_1 \rightarrow \infty$ (for $n > 2$).

2. Second family $(n = -2, w > -1)$

If $n = -2$, solutions exist only if $m = +2$ so that both G and Λ are proportional to t^{-2} . The remarkable property of this cosmology is that its scale factor and energy density are constant, the Universe does not expand, even though G and Λ have a nontrivial time dependence:

$$a(t) = \left[\frac{1}{4} (1+w) \mathcal{M} C \right]^{1/(3+3w)} = \text{const}, \quad (6.14a)$$

$$\rho(t) = \frac{1}{2\pi} \frac{1}{(1+w)} \frac{1}{C} = \text{const}, \quad (6.14b)$$

$$G(t) = C t^{-2}, \quad (6.14c)$$

$$\Lambda(t) = \frac{(3w-1)}{(1+w)} t^{-2}. \quad (6.14d)$$

(Since $\dot{a} = 0$ here, the ii -component of Einstein's equation has been checked explicitly in the present case.) The cosmology (6.14) is similar to (4.31) but contrary to the latter it has a positive energy density which confirms the general picture.

6.2.1 Third family $(n = -m \neq -2, w = -1)$

This family describes expanding or contracting Universes whose matter energy density remains constant:

$$a(t) = A t^{n/2}, \quad (6.15a)$$

$$\rho(t) = \frac{\mathcal{M}}{8\pi}, \quad (6.15b)$$

$$G(t) = C t^n, \quad (6.15c)$$

$$\Lambda(t) = -\mathcal{M} C t^n. \quad (6.15d)$$

The overall scale A is arbitrary and \mathcal{M} can have either sign.

7 Summary and Conclusion

In this paper we discussed the general framework describing the gravitational dynamics in presence of position-dependent “constants” $G(x)$ and $\Lambda(x)$ which result from RG-improving the Einstein-Hilbert action. The x -dependence of G and Λ is governed by a RG trajectory on a truncated theory space, together with a cutoff identification $k = k(x)$ relating spacetime points to RG scales. The improvement is effected by the replacement $G \rightarrow G(x)$, $\Lambda \rightarrow \Lambda(x)$ in the Lagrangian density. The resulting formalism has a certain similarity with Brans-Dicke theory, but there are also crucial differences. In particular, $G(x)$ and $\Lambda(x)$ are externally prescribed background fields in the present case since the RG equations admit no simple (local) Lagrangian description. We derived the modified Einstein field equations, and we showed that their consistency imposes certain conditions upon the scalar fields $G(x)$ and $\Lambda(x)$. The main property both theories have in common is that those scalar fields carry energy and momentum and contribute to the curvature of spacetime therefore. As for the energy-momentum tensor pertaining to G and Λ , either theory contains the piece $\Delta T_{\mu\nu}$ which results from varying $\sqrt{-g} R/G(x)$ with respect to the metric. In addition, conventional Brans-Dicke theory contains a term $\mathcal{T}_{\mu\nu}^\omega$ which is a consequence of the (almost) standard kinetic term $\propto \omega (D\phi)^2$ which Brans and Dicke postulated for the field $\phi \equiv 1/G$. This term has no immediate analog within the present framework since the RG flow is not described in a Lagrangian setting. Therefore, to be as general as possible, we included an a priori arbitrary tensor $\theta_{\mu\nu}$ in the modified Einstein equation which, together with $\Delta T_{\mu\nu}$, is to describe the 4-momentum residing in the x -dependence of G and Λ . The form of $\theta_{\mu\nu}$ is severely constrained by the consistency condition. In the extreme case of $\Lambda \equiv 0$ and a traceless energy-momentum tensor of the matter fields (“Class II”), for instance, $\theta_{\mu\nu}$ gets uniquely fixed. We found that $\theta_{\mu\nu} = \theta_{\mu\nu}^{\text{BD}}$ where $\theta_{\mu\nu}^{\text{BD}}$ equals the Brans-Dicke tensor $\mathcal{T}_{\mu\nu}^\omega$ for the exceptional value $\omega = -3/2$. From the point of view of ordinary Brans-Dicke theory, $\omega = -3/2$ amounts to the singular

limit where the Klein-Gordon equation for ϕ decouples and no longer determines or constrains $G(x)$. The reason is that precisely for $\omega = -3/2$ the kinetic term $\propto \omega (D\phi)^2$ can be absorbed into the $\sqrt{-g} R$ -term by a Weyl-rescaling of the metric. This mechanism allows us to treat $G(x)$ as an externally prescribed field then. It also provides us with a remarkably simple and efficient method for solving the in general rather complicated modified Einstein equations with $\theta_{\mu\nu} = \theta_{\mu\nu}^{\text{BD}}$. If $T_\mu{}^\mu = 0$ and either $\Lambda \equiv 0$ (“Class II”) or $\Lambda(x) \propto 1/G(x)$ (“Class IIIa”), all solutions can be obtained by Weyl-transforming solutions of the corresponding Einstein equations with constant G and Λ , which are solved much more easily, of course. The classes II and IIIa include applications which are of particular physical interest. In “small” systems such as black holes, say, the cosmological constant does not play a central role typically and may be neglected so that we are in Class II, and in all situations where the underlying RG trajectory is close to a fixed point one has $\Lambda(x) \propto 1/G(x)$ and we are in Class IIIa. The condition on the matter system, the tracelessness of $T_{\mu\nu}$, is satisfied most trivially in the vacuum, but clearly one may also think of classical radiation or a (quantum) conformal field theory.

Once we have solved the RG equations for $G(k)$ and $\Lambda(k)$, and have converted the k -dependence to a x -dependence, various strategies for exploiting this information in a dynamical context suggest themselves. In the present approach we replaced $G \rightarrow G(x)$, $\Lambda \rightarrow \Lambda(x)$ in the Lagrangian density; alternatively one could, for instance, first derive the standard Einstein equation from the classical action in the usual way, and then replace $G \rightarrow G(x)$, $\Lambda \rightarrow \Lambda(x)$ at the level of the equation of motion. In general the field equations obtained by the two methods are different. Those obtained by varying the improved action functional contain terms involving derivatives of G (the tensor $\Delta T_{\mu\nu}$) which could never arise by improving the equation of motion. As a result, in the former approach, there is energy and momentum associated to the variation of G in time and space, while this is not the case in the latter.

Thus at first sight it might seem that the two approaches lead to quite different

physical predictions so that at least one of them should be wrong. In this paper we demonstrated that this is actually not the case and that, provided both methods are applicable, they can very well lead to identical or at least qualitatively similar results. However, matching the two approaches is not straightforward. In particular, it requires either a re-interpretation of the “matter” energy-momentum tensor $T_{\mu\nu}$, or the inclusion of an appropriate θ -tensor.

In order to clarify these issues we chose Robertson-Walker cosmology as a first application because it is a typical and at the same time technically simple and transparent example. For the same reason we employed the time dependence of G and Λ arising from a RG trajectory near a fixed point in most of our examples¹¹. For the choice $\theta_{\mu\nu} = 0$ we found the following rather surprising result: If one interprets $T_{\mu\nu} + \Delta T_{\mu\nu}$ rather than $T_{\mu\nu}$ alone as the matter energy-momentum tensor, the spatially flat fixed point cosmologies obtained by both approaches coincide *exactly* if $w = 1/3$ and approximately for other values of w . Also in many other cosmologies we observed the same phenomenon. In Section 4 we discussed and explained it in detail, and in Section 5 and 6 we showed that the “contamination” of the matter energy-momentum tensor by contributions due to the x -dependence of G and Λ can be avoided if one allows for a non-zero θ -tensor. In our examples we adapted the choice $\theta_{\mu\nu} = \theta_{\mu\nu}^{\text{BD}}$, both because this tensor is the one used in ordinary Brans-Dicke theory and because of its uniqueness property mentioned above. In the classes II and IIIa, thanks to this choice, we were able to obtain strikingly simple closed-form solutions to the quite complicated differential equations governing the cosmological evolution, some of them valid for arbitrary $G(t)$ even.

The upshot of our general discussion and the analysis of the cosmological examples is that in a proper application of the improved-Lagrangian approach one should include a

¹¹We also performed analogous calculations in the “perturbative regime” where one expands in powers of k/m_{Pl} or t_{Pl}/t , respectively, so that one can see the transition from the classical to the quantum domain. Since the results are quite lengthy and not particularly instructive we do not display them here.

S_θ -term into the total action. We saw that there is a dual motivation for it: it makes the modified Einstein equation consistent, and at the same time it brings the improved-action approach closer to the improved-equation method. Even if the analogy is not complete we can think of S_θ as an analogue of the nonpolynomial potential term $\propto \Phi^4 \ln(\Phi)$ or the nonlocal kinetic term $\propto \Phi Z(\sqrt{-\partial^2}) \partial^2 \Phi$ discussed in the Introduction. While not contained in the truncation ansatz, the RG reasoning suggests that these terms should be present in Γ .

In conclusion we can say that in this paper we have gained a physical understanding of how to interpret the results from the improved action-approach and how to relate them to those obtained earlier by improving the field equations. The new approach has a much wider range of applicability than the older one, and as we now understand how to handle it properly it will be possible to apply it to situations where the improvement of the field equations makes no sense. The most important example of this kind are vacuum spacetimes in absence of a cosmological constant, the Schwarzschild black hole, for instance. They satisfy $G_{\mu\nu} = 0$, and clearly this equation is completely “blind” to a possible x -dependence of Newton’s constant. It will be interesting therefore to investigate the quantum properties of black holes within the framework developed in this paper. We shall come back to this point elsewhere [34].

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Appendix

A Improving the field equations

In this appendix we collect some of the results of ref. [18] which are needed in the main text. In [18] the approach of RG improving field equations (as opposed to solutions or actions)

has been applied to cosmology. The starting point is the standard Einstein equation without additional terms, $G_{\mu\nu} = -\Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu}$. The RG improvement consists of replacing $G \rightarrow G(t)$, $\Lambda \rightarrow \Lambda(t)$ in this equation. Specializing for a Robertson-Walker metric, one obtains the following system of coupled equations:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{1}{3}\Lambda + \frac{8\pi}{3}G\rho \quad (\text{A.1a})$$

$$\dot{\rho} + 3(1+w)\left(\frac{\dot{a}}{a}\right)\rho = 0 \quad (\text{A.1b})$$

$$\dot{\Lambda} + 8\pi\rho\dot{G} = 0. \quad (\text{A.1c})$$

Eq. (A.1a) has the form of the standard Friedmann equation with a time-dependent G and Λ inserted, eq. (A.1b) is the usual continuity equation, and eq. (A.1c) is the consistency condition resulting from the integrability condition $D^\mu[-\Lambda g_{\mu\nu} + 8\pi G T_{\mu\nu}] = 0$.

With $G(k)$ and $\Lambda(k)$ in the fixed point regime and the cutoff identification $k = \xi/t$ the time dependence of G and Λ is given by (1.21). For this time dependence, and a fixed value of the density parameter \mathcal{M} , the system (A.1) has the following unique solution:

$$a(t) = \left[\left(\frac{3}{8}\right)^2 (1+w)^4 \mathcal{M} g_* \lambda_* \right]^{1/(3+3w)} t^{4/(3+3w)} \quad (\text{A.2a})$$

$$\rho(t) = \frac{8}{9\pi} \frac{1}{(1+w)^4} \frac{1}{g_* \lambda_*} \frac{1}{t^4} \quad (\text{A.2b})$$

$$G(t) = \frac{3}{8} (1+w)^2 g_* \lambda_* t^2 \quad (\text{A.2c})$$

$$\Lambda(t) = \frac{8}{3} \frac{1}{(1+w)^2} \frac{1}{t^2}. \quad (\text{A.2d})$$

The integrability of (A.1) fixes the constant ξ according to

$$\xi^2 = \frac{8}{3(1+w)^2} \frac{1}{\lambda_*}. \quad (\text{A.3})$$

For a detailed discussion of the cosmology (A.2) we refer to [18, 22]. Further solutions of the system (A.1), in particular in the perturbative regime of the renormalization group (expansion in powers of k/m_{Pl}) can be found in [18].

B Solving the consistency condition (2.26)

In this appendix we prove that $\vartheta_{\mu\nu}^{\text{BD}}$ of eq. (2.27) is the unique tensor which satisfies the consistency condition (2.26) for *all* functions ψ and which vanishes for $\psi = \text{const.}$ (For special ψ 's further solutions might exist.)

To start with, we look for the most general solution $\vartheta_{\mu\nu}$, constructed from ψ and its derivatives, which contains no more than two derivatives. We make an ansatz

$$\begin{aligned} \vartheta_{\mu\nu} = & A(\psi) D_\mu \psi D_\nu \psi + B(\psi) g_{\mu\nu} (D\psi)^2 \\ & + C(\psi) D_\mu D_\nu \psi + D(\psi) g_{\mu\nu} D^2 \psi + g_{\mu\nu} E(\psi) \end{aligned} \quad (\text{B.1})$$

with A, B, C, \dots arbitrary functions of ψ . Upon inserting (B.1) into (2.26) combinations of those functions and their derivatives multiply various field monomials such as $D_\mu \psi D^\mu \psi D^\nu \psi$, $D^2 \psi D^\nu \psi$, etc.; since ψ is assumed arbitrary, these monomials are linearly independent and so their prefactors must vanish separately. This leads to a system of differential equations for A, B, C, \dots whose general solution can be found easily. Inserting it into (B.1) we obtain

$$\vartheta_{\mu\nu} = -\frac{3}{2} \left[D_\mu \psi D_\nu \psi - \frac{1}{2} g_{\mu\nu} (D\psi)^2 \right] + g_{\mu\nu} E(0) e^\psi \quad (\text{B.2})$$

The only free constant of integration is $E(0)$. As $\vartheta_{\mu\nu}$ must vanish for $\psi = \text{const}$ we are forced to set $E(0) = 0$, in which case $\vartheta_{\mu\nu}$ becomes equal to $\vartheta_{\mu\nu}^{\text{BD}}$.

In a second step, we write the most general solution as

$$\vartheta_{\mu\nu} = \vartheta_{\mu\nu}^{\text{BD}} + \widehat{\vartheta}_{\mu\nu}. \quad (\text{B.3})$$

Since $\vartheta_{\mu\nu}^{\text{BD}}$ is a special solution to the inhomogeneous equation (2.26), $\widehat{\vartheta}_{\mu\nu}$ is the general solution of the homogeneous equation

$$D^\mu \widehat{\vartheta}_{\mu\nu} + \left(\widehat{\vartheta}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \widehat{\vartheta}_\alpha{}^\alpha \right) D^\mu \psi = 0. \quad (\text{B.4})$$

It simplifies when we rewrite it in terms of $\widehat{\theta}_{\mu\nu} \equiv \widehat{\vartheta}_{\mu\nu}/(8\pi G)$ which enters the analogous decomposition $\theta_{\mu\nu} = \theta_{\mu\nu}^{\text{BD}} + \widehat{\theta}_{\mu\nu}$:

$$D^\mu \widehat{\theta}_{\mu\nu} = \frac{1}{2} \widehat{\theta}_\alpha{}^\alpha D_\nu \psi. \quad (\text{B.5})$$

We assume that $\widehat{\theta}_{\mu\nu}$ is generated by the action $\widehat{S}_\theta[g_{\mu\nu}, \psi]$:

$$\widehat{\theta}^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \widehat{S}_\theta}{\delta g_{\mu\nu}}. \quad (\text{B.6})$$

The action \widehat{S}_θ is invariant under general coordinate transformations if $g_{\mu\nu}$ and ψ transform as a tensor and a scalar, respectively. Therefore, by the same argument which lead to (2.19),

$$D^\mu \widehat{\theta}_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta \widehat{S}_\theta}{\delta \psi} D_\nu \psi. \quad (\text{B.7})$$

Using (B.7) in (B.5) for $D_\mu \psi \neq 0$ the problem boils down to finding the general solution of

$$\left[g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} - \frac{\delta}{\delta \psi} \right] \widehat{S}_\theta[g, \psi] = 0. \quad (\text{B.8})$$

Eq. (B.8) is the infinitesimal form of the invariance condition

$$\widehat{S}_\theta[e^\alpha g_{\mu\nu}, \psi - \alpha] = \widehat{S}_\theta[g_{\mu\nu}, \psi], \quad (\text{B.9})$$

where $\alpha(x)$ is an arbitrary parameter. Eq. (B.9) tells us that \widehat{S}_θ is not a functional of $g_{\mu\nu}$ and ψ separately but only of the combination $e^\psi g_{\mu\nu} \equiv \gamma_{\mu\nu}$. Writing $\widehat{S}_\theta[g, \psi] = F[e^\psi g] \equiv F[\gamma]$, the corresponding energy-momentum tensor reads

$$\widehat{\theta}^{\mu\nu}[g, \psi] = e^{3\psi} \left(\frac{2}{\sqrt{-\gamma}} \frac{\delta F[\gamma]}{\delta \gamma_{\mu\nu}} \right) [\gamma_{\alpha\beta} = e^\psi g_{\alpha\beta}]. \quad (\text{B.10})$$

At this point the condition that $\theta_{\mu\nu}$ must vanish for $\psi = \text{const}$ plays an important role. Since $\theta_{\mu\nu}^{\text{BD}}$ does have this property it follows that $\widehat{\theta}_{\mu\nu}$ must vanish separately for constant ψ . However, for $F[\gamma]$ an arbitrary functional, the tensor (B.10) does not in general vanish for $\psi = \text{const}$. Thus we see that tensors of the form (B.10) are not admissible, except when $F[\gamma] = 0$, $\widehat{\theta}_{\mu\nu} = 0$. This concludes our proof that the unique, identical solution of (2.26) is the Brans-Dicke-type tensor $\vartheta_{\mu\nu}^{\text{BD}}$.

C Eliminating the ii -components

In this appendix we prove that, under the condition $\dot{a} \neq 0$, every set of functions $a(t)$, $\rho(t)$, $p(t)$ which satisfies the 00-component of the modified Einstein equation, the consistency condition, and the continuity equation (3.4) automatically satisfies the ii -components of the modified Einstein equation as well.

The demonstration proceeds as follows: Defining the tensor $T'_{\mu\nu}$ by

$$\begin{aligned} 8\pi\overline{G}T'_{\mu\nu} &\equiv -g_{\mu\nu}\Lambda + 8\pi G(T_{\mu\nu} + \Delta T_{\mu\nu} + \theta_{\mu\nu}) \\ &= -g_{\mu\nu}\Lambda + 8\pi G T_{\mu\nu} + \Delta t_{\mu\nu} + \vartheta_{\mu\nu} \end{aligned} \quad (\text{C.1})$$

with an arbitrary, fixed constant \overline{G} , Einstein's equation (2.9) assumes the standard form

$$G_{\mu\nu} = 8\pi\overline{G}T'_{\mu\nu}. \quad (\text{C.2})$$

For symmetry reasons, $T'^{\mu\nu}$ has the structure $T'^{\mu\nu} = \text{diag}[-\rho', p', p', p']$ with ρ' and p' depending on t only. In $T'_{\mu\nu}$ -language, since \overline{G} is constant, the consistency condition (2.20) is equivalent to the statement that $T'_{\mu\nu}$ has vanishing covariant divergence, i.e. that $D_\mu T'^{\mu\nu} = 0$, or

$$\dot{\rho}' + 3H(\rho' + p') = 0. \quad (\text{C.3})$$

Moreover, the 00- and ii -components of (C.2) assume the same form as in standard cosmology without a cosmological constant, albeit with a complicated equation of state $p' = p'(\rho')$. The 00-component is

$$H^2 + \frac{K}{a^2} = \frac{8\pi}{3}\overline{G}\rho', \quad (\text{C.4a})$$

and the ii -component reads

$$H^2 + 2\left(\frac{\ddot{a}}{a}\right) + \frac{K}{a^2} = -8\pi\overline{G}p'. \quad (\text{C.4b})$$

If functions $\rho'(t)$, $p'(t)$, $a(t)$ satisfy (C.3) and (C.4a), they also satisfy (C.4b).

This can be seen by differentiating (C.4a) with respect to t ,

$$2 \left(\frac{\ddot{a}}{a} \right) H - 2 H^3 - 2 \frac{K}{a^2} H = \frac{8\pi}{3} \overline{G} \dot{\rho}',$$

and substituting (C.3) for $\dot{\rho}'$:

$$2 \left(\frac{\ddot{a}}{a} \right) H - 2 H^3 - 2 \frac{K}{a^2} H = -8\pi \overline{G} H (\rho' + p').$$

This equation implies

$$2 H^2 + 2 \frac{K}{a^2} = 2 \left(\frac{\ddot{a}}{a} \right) + 8\pi \overline{G} (\rho' + p'), \quad (\text{C.5})$$

provided $H \neq 0$. Inserting (C.5) into (C.4a) according to

$$\begin{aligned} 8\pi \overline{G} \rho' &= \left[2 H^2 + 2 \frac{K}{a^2} \right] + \left[H^2 + \frac{K}{a^2} \right] \\ &= 2 \left(\frac{\ddot{a}}{a} \right) + 8\pi \overline{G} (\rho' + p') + \left[H^2 + \frac{K}{a^2} \right], \end{aligned}$$

leads precisely to the ii -component (C.4b), which is thus seen to be a consequence of the 00-component and the continuity equation if the scale factor $a(t)$ is not constant, i. e. if $H \neq 0$.

Thus, rather than (C.4a) and (C.4b), one may use the 00-component (C.4a) and the conservation law for $T'_{\mu\nu}$, (C.3), as independent equations. Returning now to the original formulation without the primed quantities, it is clear that the integrability condition, $D_\mu T'^{\mu\nu} = 0$, is nothing but the off-shell consistency condition (2.20) or, more explicitly, (3.13). (In deriving it we used indeed that the divergence of (C.1) must be zero and assumed that the ordinary continuity equation $D_\mu T^{\mu\nu} = 0$ is valid.) This shows that if a cosmology with $a \neq \text{const}$ satisfies the 00-component of Einstein's equation, eq. (3.8a), the consistency condition (3.13) and the ordinary continuity condition (3.4), then, for any equation of state, it also satisfies the ii -component of Einstein's equation.

References

- [1] C. Brans and R.H. Dicke, Phys. Rev. 124 (1961) 925.
- [2] O. Bertolami et al., Phys. Lett. B 311 (1993) 27;
O. Bertolami and J. Garcia-Bellido, Int. J. Mod. Phys. D 5 (1996) 363 .
- [3] C. Wetterich, Phys. Lett. B 301 (1993) 90.
- [4] For a review see: J. Berges, N. Tetradis and C. Wetterich, Phys. Rep. 363 (2002) 223; C. Wetterich, Int. J. Mod. Phys. A 16 (2001) 1951.
- [5] M. Reuter and C. Wetterich, Nucl. Phys. B 417 (1994) 181, Nucl. Phys. B 427 (1994) 291, Nucl. Phys. B 391 (1993) 147, Nucl. Phys. B 408 (1993) 91;
M. Reuter, Phys. Rev. D 53 (1996) 4430, Mod. Phys. Lett. A 12 (1997) 2777.
- [6] M. Reuter, Phys. Rev. D 57 (1998) 971 and hep-th/9605030.
- [7] D. Dou and R. Percacci, Class. Quant. Grav. 15 (1998) 3449.
- [8] M. Carfora and K. Piotrkowska, Phys. Rev. D 52 (1995) 4393;
T. Buchert and M. Carfora, gr-qc/0101070;
T. Buchert and M. Carfora, Phys. Rev. Lett. 90 (2003) 031101.
- [9] For a review see: C. Bagnuls and C. Bervillier, Phys. Rep. 348 (2001) 91;
T.R. Morris, Prog. Theor. Phys. Suppl. 131 (1998) 395.
- [10] C. Wetterich, Gen. Rel. Grav. 30 (1998) 159.
- [11] O. Lauscher and M. Reuter, Phys. Rev. D 65 (2001) 025013 and hep-th/0108040.
- [12] M. Reuter and F. Saueressig, Phys. Rev. D 65 (2002) 065016 and hep-th/0110054.

- [13] O. Lauscher and M. Reuter, *Class. Quant. Grav.* 19 (2002) 483 and hep-th/0110021;
Phys. Rev. D 66 (2002) 025026 and hep-th/0205062;
Int. J. Mod. Phys. A 17 (2002) 993 and hep-th/0112089.
- [14] R. Percacci and D. Perini, *Phys. Rev. D* 67 (2003) 081503; *Phys. Rev. D* 68 (2003) 044018 ; D. Perini, hep-th/0305053.
- [15] M. Reuter and F. Saueressig, *Phys. Rev. D* 66 (2002) 125001 and hep-th/0206145.
- [16] G. Veneziano, *Mod. Phys. Lett. A* 4 (1989) 695;
T. Taylor and G. Veneziano, *Phys. Lett. B* 228 (1990) 210.
- [17] B. L. Nelson, P. Panangaden, *Phys. Rev. D* 25 (1982) 1019.
- [18] A. Bonanno and M. Reuter, *Phys. Rev. D* 65 (2002) 043508 and hep-th/0106133.
- [19] A. Bonanno and M. Reuter, *Phys. Lett. B* 527 (2002) 9 and astro-ph/0106468;
Int. J. Mod. Phys. D, in press, and astro-ph/0210472;
E. Bentivegna, A. Bonanno and M. Reuter, astro-ph/0303150.
- [20] A. Bonanno, G. Esposito and C. Rubano, hep-th/0303154.
- [21] A. Bonanno and M. Reuter, *Phys. Rev. D* 62 (2000) 043008 and hep-th/0002196;
Phys. Rev. D 60 (1999) 084011 and gr-qc/9811026.
- [22] D. Kalligas, P. Wesson and C.W.F. Everitt, *Gen. Rel. Grav.* 24 (1992)351;
for a similar cosmology in a Brans-Dicke framework see:
O. Bertolami and P.J. Martins, *Phys. Rev. D* 61 (2000) 064007.
- [23] W. Souma, *Prog. Theor. Phys.* 102 (1999) 181.
- [24] S. Weinberg in *General Relativity, an Einstein Centenary Survey*, S.W. Hawking and W. Israel (Eds.), Cambridge University Press (1979); S. Weinberg, hep-th/9702027.

- [25] P. Forgács and M. Niedermaier, hep-th/0207028;
M. Niedermaier, JHEP 0212 (2002) 066; Nucl. Phys. B 673 (2003) 131.
- [26] O. Bertolami, Nuovo Cim. B 93 (1986) 36.
- [27] I.L. Shapiro and J. Solà, Phys. Lett. B 475 (2000) 236; JHEP 02 (2002) 006;
I.L. Shapiro, J. Solà, C. España-Bonet and P. Ruiz-Lapiente, Phys. Lett. B 574 (2003) 149.
- [28] M. Reuter and C. Wetterich, Phys. Lett. B 188 (1987) 38.
- [29] C. Wetterich, Nucl. Phys. B 302 (1988) 668; Astron. Astrophys. 301 (1995) 321.
- [30] P. J. Peebles and B. Ratra, Astrophys. J. Lett. 325 (1988) L 17;
B. Ratra and P. J. Peebles, Phys. Rev. D 37 (1988) 3406.
- [31] For a review and a comprehensive list of references see: V. Faraoni, E. Gunzig and P. Nardone, Fund. Cosmic Phys. 20 (1999) 121.
- [32] Y. Fujii and K. Maeda, *The scalar-tensor theory of gravitation*, Cambridge University Press (2002).
- [33] M. Blagojević, *Gravitation and Gauge Symmetries*, IOP Publishing, Bristol (2002).
- [34] M. Reuter and H. Weyer, in preparation